MATHEMATICS

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Class XII

VECTOR

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VECTORS

1. DEFINITIONS :

A VECTOR may be described as a quantity having both magnitude & direction. A vector is generally represented by a directed line segment, say $\vec{AB}$. A is called the initial point & B is called the terminal point . The magnitude of vector $\vec{AB}$ is expressed by $\left|\vec{AB}\right|$.

The modulus, or magnitude, of a vector is the positive number which is the measure of its length. The modulus of the vector **a** is sometimes denoted by |a|, and sometimes by the corresponding symbol a in italics. The vector which has the same modulus as a, but the opposite direction, is called the negative of **a**, and is denoted by –**a**.

Let the vectors **a, b** be represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Then the inclination of the vectors, or the angle between them, is defined as that angle AOB which does not exceed π. Thus if θ denote this inclination, $0 \le \theta \le \pi$. When the inclination is $\pi/2$. the vectors are said to be perpendicular; when it is 0 or π they are parallel.

ZERO VECTOR a vector of zero magnitude i.e.which has the same initial & terminal point, is called a **ZERO VECTOR**. It is denoted by **o**.

UNIT VECTOR a vector of unit magnitude in direction of a vector $\vec{a}$ is called unit vector along $\vec{a}$ and is denoted by $\hat{a}$ symbolically $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$.

EQUAL VECTORS two vectors **a** & **b** are said to be equal if they have the same magnitude, direction & represent the same physical quantity. This is denoted symbolically by **a** = **b**.

COLLINEAR VECTORS two vectors are said to be collinear if their directed line segments are parallel disregards to their direction. Collinear vectors are also called **PARALLEL VECTORS**. If they have the same direction they are named as like vectors otherwise unlike vectors.

Symbolically, two non zero vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if, $\vec{a} = K\vec{b}$, where $K \in R$

COPLANAR VECTORS a given number of vectors are called coplanar if their directed line segments are all parallel to the same plane. Note that "Two Vectors Are Always Coplanar".

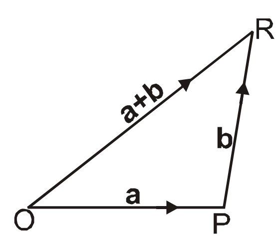
Vectors as defined above are usually called free vectors, since the value of such a vector depends only on its length and direction and is independent of its position in space. A single free vector cannot therefore completely represent the effect of a localized vector quanity, such as a force acting on a rigid body. This effect depends on the line of action of the force; and it wil be shown later that two free vectors are neccessary for its specification.

2. VECTOR ADDITION :

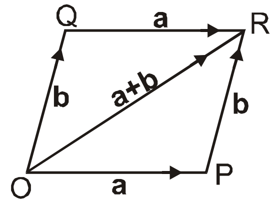
Addition and Subtraction of Vectors. The manner in which the vector quantities of mechanics and physics are compounded is expressed by the triangle law of addition, which may be stated as follows :

If three points O, P, R are chosen so that $\overrightarrow{OP}$ = **a** and $\overrightarrow{PR}$ = **b** then the vector $\overrightarrow{OR}$ is called the (vector) sum or resulatant of **a** and **b**.

Denoting this resultant by c, we write **c** = **a** + **b**,



borrowing the sign + from algebra, and using the term vector addition for the process by which the resultant **c** is obtained from the component vectors **a** and **b**. The above definition is not an arbitrary mathematical assumption. It is an expression of the way in which the vector quantities of physics and mechanics are compounded. We see also that the sum of two vectors **a** = $\overrightarrow{OP}$ and **b** = $\overrightarrow{OQ}$ is the vector $\overrightarrow{OR}$ determined by the diagonal of the parallelogram of which OP and OQ are sides.



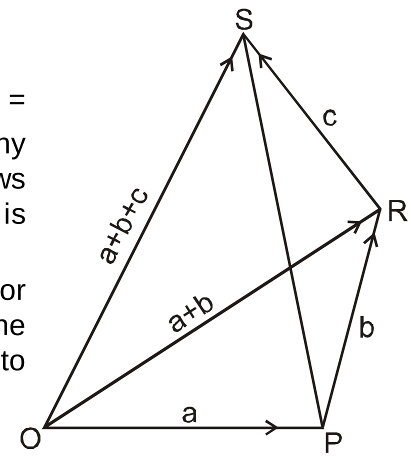
The triangle law of addition is identical with the parallelogram law

Further since $\overrightarrow{QR} = \overrightarrow{OP}$ = **a** it follows that **b** + **a** = $\overrightarrow{OQ} + \overrightarrow{QR} = \overrightarrow{OR}$,

showing that **b** + **a** = **a** + **b** = **r** (say).

Again, we may add to this another vector **c** = $\overrightarrow{RS}$, obtaining the result

$\overrightarrow{OS}$ = **r** + **c** = (**a** + **b**) + **c** = **c** + (**a** + **b**).



But a glance at fig. shows that this vector is also $\overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{PS}$ = **a** + (**b** + **c**) = (**b** + **c**) + **a**, and the argument obviously holds for any number of vectors. Hence the commutative and associative laws hold for the addition of any number of vectors. The sum is independent of the order and the grouping of the terms.

We have already stated that –**b** is to be understood as the vector which has the same length as **b**, but the oposite direction. The subtraction of **b** from **a** is to be understood as the addition of –**b** to **a**. We denote this by

a – **b** = **a** + (–**b**),

borrowing the – sign from algebra. Thus to subtract the vector **b** from **a**, reverse the direction of **b** and add. **a** – **b** is represented by the (undrawn) diagonal QP; for **a** – **b** = $\overrightarrow{QR} + \overrightarrow{RP} = \overrightarrow{QP}$.

For the particular case in which **b** = **a** we have **a** – **a** = **0**.

All zero vectors are regarded as equal irrespective of direction. Indeed we may say that the direction of a zero vector is arbitrary. Vectors other than the zero vector are called proper vectors.

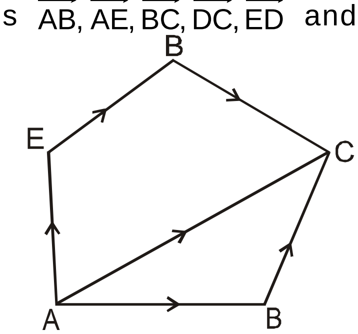
For any vectors **a** and **b** we have the following inequality

| **a** + **b** | ≤ | **a** | + | **b** |

(the triangle inequality), geometrically expressing the fact that in a triangle the sum of its two sides is greater than the third side if the vectors are not parallel. This inequality is obviously valid for any number of vectors :

| **a** + **b** +..... + ***l*** | ≤ | **a** | + | **b** | +...... + | ***l*** |.

Ex.1 ABCDEis a pentagon. Prove that the resultant of the forces $\overrightarrow{AB}, \overrightarrow{AE}, \overrightarrow{BC}, \overrightarrow{DC}, \overrightarrow{ED}$ and $\overrightarrow{AC}$ is $3\overrightarrow{AC}$.



Sol. Let R be the resultant force.

$\therefore$ R = $\overrightarrow{AB} + \overrightarrow{AE} + \overrightarrow{BC} + \overrightarrow{DC} + \overrightarrow{ED} + \overrightarrow{AC}$

$\therefore$ R = $(\overrightarrow{AB} + \overrightarrow{AE}) + (\overrightarrow{AE} + \overrightarrow{ED} + \overrightarrow{DC}) + \overrightarrow{AC}$

= $\overrightarrow{AC} + \overrightarrow{AC} + \overrightarrow{AC}$

=3$\overrightarrow{AC}$. Hence proved.

3. MULTIPLICATION OF VECTOR BY SCALARS :

If $\vec{a}$ is a vector & m is a scalar, then m$\vec{a}$ is a vector parallel to $\vec{a}$ whose modulus is $|m|$ times that of $\vec{a}$. This multiplication is called **Scalar Multiplication**. If $\vec{a}$ & $\vec{b}$ are vectors & m, n are scalars, then :

$m(\vec{a}) = (\vec{a})m = m\vec{a}$ $\qquad$ $m(n\vec{a}) = n(m\vec{a}) = (mn)\vec{a}$

$(m+n)\vec{a} = m\vec{a} + n\vec{a}$ $\qquad$ $m(\vec{a}+\vec{b}) = m\vec{a} + m\vec{b}$

4. POSITION VECTOR

With an assigned point O as origin, the position of any point P is specified uniquely by the vector $\overrightarrow{OP}$, which is called the position vector of P relative to O. It will be found convenient to denote the position vectors of the points A, B, C,....Z by the corresponding small Clarendon symbols **a, b, c,...z**. With this notation the vector AB is **b** – **a**. For

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}. \qquad (1)$$

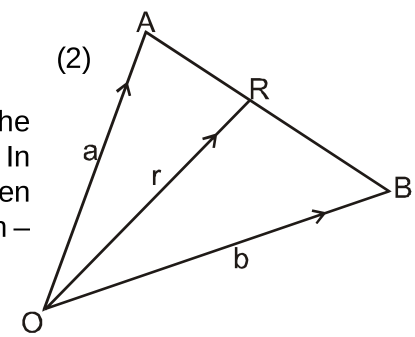
A point with position vector **r** is often referred to as the point **r**.

Section Formula :

If $\vec{a}$ & $\vec{b}$ are the position vectors of two points A & B then the p.v. of a point which divides AB in the ratio m : n is given by : $\vec{r} = \frac{n\vec{a} + m\vec{b}}{m+n}$. Note p.v. of mid point of AB = $\frac{\vec{a}+\vec{b}}{2}$.

Let A, B be the two points, and **a, b** their position vectors relative to the origin O. Then the position vector of the point R which divides AB in the ratio m : n, may be found in terms of **a** and **b**. For, since $n\overrightarrow{AR} = m\overrightarrow{RB}$, it follows that n(**r** – **a**) = m(**b** – **r**)

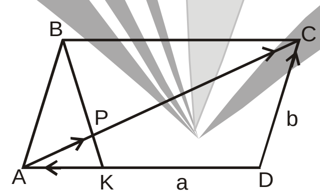
whence $$\mathbf{r} = \frac{n\mathbf{a} + m\mathbf{b}}{m+n}. \qquad (m + n \neq 0) \qquad (2)$$



This is the required expression for the position vector of R. The reasoning holds whether the ratio m : n is positive or negative. In the latter case R is outside the segment AB. If the ratio lies between o and –1, R is outside AB and nerer to A. If the ratio lies between –1 and –∞,, R is outside AB and nearer to B.

For the particular case in which m = n, the above formula given $\frac{1}{2}$(**a** + **b**) for the position vector of the mid-point of AB.

Ex.2 The side AD of the parallelogram ABCD is divided into n equal parts and the first division point (point K) is joined to the vertex B (Fig.). Find the parts into which the diagonal AC is divided by the half–line BK.



Sol. Let $\overrightarrow{DC}$ = **b**, $\overrightarrow{DA}$ = **a**, and $\overrightarrow{AP} = \alpha\overrightarrow{AC}$. We express the vector $\overrightarrow{AP}$ in terms of the vectors **a** and **b** in two ways: (1) $\overrightarrow{AP} = \alpha\overrightarrow{AC} = \alpha(\mathbf{b} - \mathbf{a}) = \alpha\mathbf{b} - \alpha\mathbf{a}$; (2) $\overrightarrow{AP} = \overrightarrow{AK} + \overrightarrow{AP} = -\frac{1}{n}\mathbf{a} + \alpha\overrightarrow{KB} = -\frac{1}{n}\mathbf{a} + \alpha(\frac{1}{n}\mathbf{a} + \mathbf{b}) = \frac{\alpha - 1}{n}\mathbf{a} + \alpha\mathbf{b}$ ($\overrightarrow{KP} = a\overrightarrow{KB}$, since ΔAPK ~ ΔBPC).

Since only one representation of a vector in terms of two noncollinear vectors is possible, we have: $\frac{\alpha-1}{n} = -\alpha$, whence $\alpha = \frac{1}{n+1}$. This means that $\overrightarrow{AP} = \frac{1}{n+1}\overrightarrow{AC}$, and then , as it is easy to see AP : PC = 1 : n.

5. LINEAR COMBINATIONS :

Given a finite set of vectors $\vec{a}, \vec{b}, \vec{c}, ...$ then the vector $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c} + ...$ is called a linear combination of $\vec{a}, \vec{b}, \vec{c}, ...$ for any x, y, z ...... $\in$ R . We have the following results :

(a) If $\vec{a}, \vec{b}$ are non zero, non–collinear vectors then $x\vec{a} + y\vec{b} = x'\vec{a} + y'\vec{b} \Rightarrow x = x' ; y = y'$

(b) **FUNDAMENTAL THEOREM :** Let $\vec{a}, \vec{b}$ be non zero , non collinear vectors . Then any vector $\vec{r}$ coplanar with $\vec{a}, \vec{b}$ can be expressed uniquely as a linear combination of $\vec{a}, \vec{b}$ i.e. There exist some unique x,y $\in$ R such that $x\vec{a} + y\vec{b} = \vec{r}$.

(c) If $\vec{a}, \vec{b}, \vec{c}$ are non–zero, non–coplanar vectors then :

$$x\vec{a} + y\vec{b} + z\vec{c} = x'\vec{a} + y'\vec{b} + z'\vec{c} \Rightarrow x = x' \ , \ y = y' \ , \ z = z'$$

(d) **FUNDAMENTAL THEOREM IN SPACE :** Let $\vec{a}, \vec{b}, \vec{c}$ be non–zero, non–coplanar vectors in space. Then any vector $\vec{r}$, can be uniquily expressed as a linear combination of $\vec{a}, \vec{b}, \vec{c}$ i.e. There exist some unique x,y $\in$ R such that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{r}$.

(e) If $\vec{x}_1, \vec{x}_2, ...\vec{x}_n$ are n non zero vectors, & k_1, k_2,.....k_n are n scalars & if the linear combination $k_1\vec{x}_1 + k_2\vec{x}_2 + ... k_n\vec{x}_n = 0 \Rightarrow k_1 = 0, k_2 = 0 ..... k_n = 0$ then we say that vectors $\vec{x}_1, \vec{x}_2, ...\vec{x}_n$ are Linearly Independent Vectors .

(f) If $\vec{x}_1, \vec{x}_2, ...\vec{x}_n$ are not Linearly Independent then they are said to be Linearly Dependent vectors . i.e. if $k_1\vec{x}_1 + k_2\vec{x}_2 + ... + k_n\vec{x}_n = 0$ & if there exists at least one $k_r \neq 0$ then $\vec{x}_1, \vec{x}_2, ...\vec{x}_n$ are said to be Linearly Dependent .

Note 1 : If $k_r \neq 0$; $k_1\vec{x}_1 + k_2\vec{x}_2 + k_3\vec{x}_3 + ... + k_r\vec{x}_r + ... + k_n\vec{x}_n = 0$

$$-k_r\vec{x}_r = k_1\vec{x}_1 + k_2\vec{x}_2 + ... + k_{r-1}.\vec{x}_{r-1} + k_{r+1}.\vec{x}_{r+1} + ... + k_n\vec{x}_n$$

$$-k_r\frac{1}{k_r}\vec{x}_r = k_1\frac{1}{k_r}\vec{x}_1 + k_2\frac{1}{k_r}\vec{x}_2 + ... + k_{r-1}.\frac{1}{k_r}\vec{x}_{r-1} + ... + k_n\frac{1}{k_r}\vec{x}_n$$

$$\vec{x}_r = c_1\vec{x}_1 + c_2\vec{x}_2 + ... + c_{r-1}\vec{x}_{r-1} + c_r\vec{x}_{r-1} + ... + c_n\vec{x}_n$$

i.e. $\vec{x}_r$ is expressed as a linear combination of vectors .

$\vec{x}_1 \ , \ \vec{x}_2 \ , ... \vec{x}_{r-1}, \vec{x}_{r+1}, ... \vec{x}_n$

Hence $\vec{x}_r$ with $\vec{x}_1, \vec{x}_2, ... \vec{x}_{r-1}, \vec{x}_{r+1} .... \vec{x}_n$ forms a linearly dependent set of vectors .

Note 2 :

(i) If $\vec{a}$ = 3i + 2j + 5k then $\vec{a}$ is expressed as a Linear Combination of vectors i, j, k. Also , $\vec{a}$, i, j, k form a linearly dependent set of vectors . In general , every set of four vectors is a linearly dependent system.

(ii) i , j , k are Linearly Independent set of vectors. For
$K_1 i + K_2 j + K_3 k = 0 \Rightarrow K_1 = 0 = K_2 = K_3$.

(iii) Two vectors $\vec{a}$ & $\vec{b}$ are linearly dependent $\Rightarrow$ $\vec{a}$ is parallel to $\vec{b}$ i.e. $\vec{a} \times \vec{b} = 0 \Rightarrow$ linear dependence of $\vec{a}$ & $\vec{b}$. Conversely if $\vec{a} \times \vec{b} \neq 0$ then $\vec{a}$ & $\vec{b}$ are linearly independent .

(iv) If three vectors $\vec{a}, \vec{b}, \vec{c}$ are linearly dependent, then they are coplanar i.e. $[\vec{a}, \vec{b}, \vec{c}] = 0$, conversely, if $[\vec{a}, \vec{b}, \vec{c}] \neq 0$, then the vectors are linearly independent .

Ex.3 Show that the points whose position vectors are $\vec{a}+2\vec{b}+2\vec{c}, 3\vec{a}+2\vec{b}+\vec{c}, 2\vec{a}+2\vec{b}+3\vec{c}$ are collinear.

Sol. Let the given points be A, B and C andO be the point of reference.

Then $\overrightarrow{OA}=\vec{a}+2\vec{b}+5\vec{c}, \overrightarrow{OB}=3\vec{a}+2\vec{b}+\vec{c}$

and $\overrightarrow{OC}=2\vec{a}+2\vec{b}+3\vec{c}$

Let us assume that *l*, m, n be three scalar quantities, such that

$\ell\overrightarrow{OA}+m\overrightarrow{OB}+n\overrightarrow{OC}$ ...(i)

where ℓ + m + n = 0 ...(ii)

Now $\ell(\vec{a}+2\vec{b}+5\vec{c})+m(3\vec{a}+2\vec{b}+\vec{c})+n(2\vec{a}+2\vec{b}+3\vec{c})=0$

or, $(\ell + 3m + 2n)\vec{a} + (2\ell + 2m + 2n)\vec{b} + (5\ell + m + 3n)\vec{c} = 0$

$= O\vec{a}+O\vec{b}+O\vec{c}\cdot$

Comparing the coefficients of $\vec{a}, \vec{b}, \vec{c}$ on both sides, we get

$\ell + 3m + 2n = 0,\ 2\ell + 2m + 2n = 0,\ 5\ell + m + 3n = 0$

or, $\ell + m + n + 2m + n = 0$

$2(\ell + m + n) = 0, (\ell + m + n) + 4\ell + 2n = 0$

or, 2m + n = 0, $4\ell + 2n = 0$ from (ii)

or, $\ell = \frac{-1}{2}n, m = \frac{-1}{2}n$ which satisfy (ii)

Hence the condition of collinearly (i) and (ii) are satisfied. Hence the given points are collinear.

Ex.4 Examine if $\vec{i}-3\vec{j}+2\vec{k}, 2\vec{i}-4\vec{j}-\vec{k}$ and $3\vec{i}+2\vec{j}-\vec{k}$ are linearly independent or dependent.

Sol. If the vectors are linearly dependent,

$\ell(\vec{i}-3\vec{j}+2\vec{k})+m(2\vec{i}-4\vec{j}-\vec{k})+n(3\vec{i}+2\vec{j}-\vec{k})=\vec{0}$

Where ℓ + m + n are scalars not all zero.

$\Rightarrow$ $\ell + 2n + 3n = 0$ ...(i)

$-3\ell - 4m + 2n = 0$ ...(ii)

$2\ell - m - n = 0$ ...(iii)

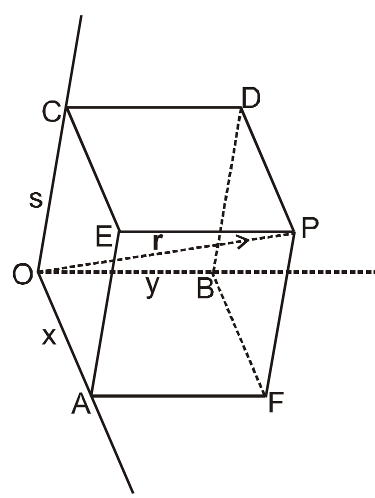
from (i) and (ii)

$\Rightarrow \frac{\ell}{16}=\frac{m}{11}=\frac{n}{2} = k$ say

$\therefore$ $\ell = 16k, m = -11k, n = 2k$

These ℓ, m, n do not satisfy (iii) and hence the given system is linearly independent.

6. RESOLUTION OF A VECTOR



Any vector **r** can be expressed as the sum of three others, parallel to any three non-coplanar vectors. Let **a, b, c** be unit vectors in the three given noncoplanar directions. With any point O as origin take $\overrightarrow{OP}$ = **r**. and on OP as diagonal construct a parallelepiped with edges OA, OB,. OC parallel to **a, b, c** respectively. Then there are real numbers x, y, z such that $\overrightarrow{OA}$ = x**a**, $\overrightarrow{OB}$ = y**b**, $\overrightarrow{OC}$ = z**c**. The number x is positive or negative accordina as $\overrightarrow{OA}$ has the same direction as **a** or the opposite directionl; and similarly for y and z. Thus the given vector is expressible as the sum

r = $\overrightarrow{OA}+\overrightarrow{AF}+\overrightarrow{FP}=\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}$

= x**a** + y**b** + z**c**.

We call **r** the resultant of the three vectors x**a**, y**b**, z**c**, which are the component vectors of **r**; while x, y, z are the components of **r** for the given directions. This resolution of **r** is unique, because only one parallelepiped can be constructed on OP as diagonal with edges parallel to the given directions. Hence, if two vectors are equal their corresponding components are equal, Conversely, if the corresponding components of two vectors are qual, the vectors are equal.

Given several vectors ri, (i = 1, 2,..., n), these may be resolved into component vectors i the given directions, and expressed in the form ri = x_i a + y_i b + z_i c. Their sum is then $\mathbf{r_i} = x_i\mathbf{a} + y_i\mathbf{b} + z_i\mathbf{c}$. Their sum is then

$$\Sigma\mathbf{r}_i = \Sigma(x_i\mathbf{a} + y_i\mathbf{b} + z_i\mathbf{c})$$
$$(\Sigma x_i)\mathbf{a} + (\Sigma y_i)\mathbf{b} + (\Sigma z_i)\mathbf{c},$$

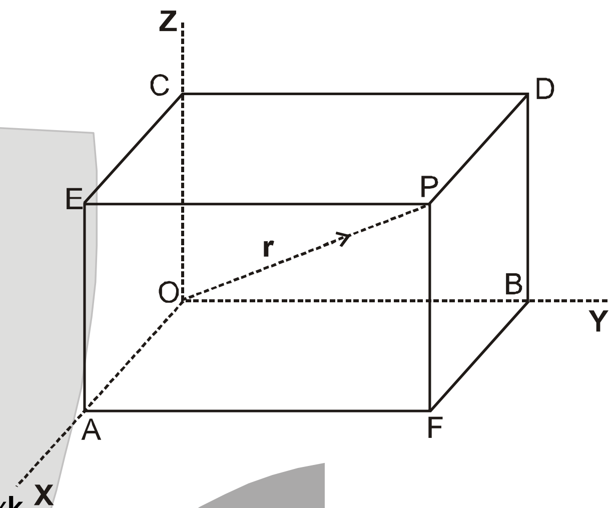
showing that vectors may be copmpounded by adding their corresponding components.

We remark that, if **r** is regarded as the position vector of the point P relative to O, then x, y, z are the Cartesian coordinates of P with respect to axes through O in the given directions.

Rectangular resolution of a vector. The most important case of resolution of vectors is that in which the three directions are mutually perpendicular. The right handed system of directions.

OX, OY, OZ

represented in the figure is found most convenient; OY and OZ are in the plane of the paper, and OX perpendicular to it pointing toward the reader. to an observer at the origin O, righthanded rotations about the axes OX, OY, OZ are form Y to Z, Z to X and X to Y respectively. The unit vectors parallel to thse axes are denoted by **i, j, k**; and the parallelepiped on OP as diagonal, with edges OA, OB, OC parallel to **i, j, k** is now rectangular. As above we may write



$$\overrightarrow{OA} = x\mathbf{i}, \overrightarrow{OB} = y\mathbf{j}, \overrightarrow{OC} = x\mathbf{k},$$

the number x being positive or negative according as $\overrightarrow{OA}$ has the same direction as **i**, or the opposite direction; and similarly for y and z. Thus the given vector is expressible in the form

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \qquad (1)$$

The component vectors x**i**, y**j**, z**k** are in this case the orthogonal projections of **r** on the directions of **i, j, k**; and x, y, z are the rectangular compontnts of **r** for these directions. The latter are often called the resolutes, or resolved parts, of **r**.

If α, β, γ are the angles which $\overrightarrow{OP}$ makes with the axes, cos α, cos β, cos γ are called the direction cosines of $\overrightarrow{OP}$, and are frequently denoted by *l*, m, n. Clearly

$$x = r\cos\alpha = lr, \quad y = r\cos\beta = mr, \quad z = r\cos\gamma = nr, \qquad (2)$$

so that the resolute of a vector in a directio, inclined at an angle θ to it, is obtained by multiplying the modulus of the vector by cos θ. the unit vector in the direction of **r** is

$$\frac{1}{r}\mathbf{r} = (\cos\alpha)\,\mathbf{i} + (\cos\beta)\mathbf{j} + (\cos\gamma)\mathbf{k} \qquad (3)$$

Thus the coeffieients of **i, j, k** in the rectangular resolution of a unit vector are the direction cosines of that vector.

It is clear from fig. and the theorem of Pythogoras that $OP^2 = OA^2 + AP^2 = OA^2 + AF^2 + FP^2$,

so that $$r^2 = x^2 + y^2 + z^2.$$

Consequenctly the square of the modulus of a vector is equal to the sum of the squares of its rectangular components. Dividing both sides of this equation by r^2 we find

$$1 + l^2 + m^2 + n^2,$$

showing that the sum of the squares of the direction cosines is equal to unity.

The sum of several vectors $\mathbf{r}_i$ is expressible in the form

$$\Sigma\mathbf{r}_i = (\Sigma x_i)\mathbf{i} + (\Sigma y_i)\mathbf{j} + (\Sigma z_i)\mathbf{k} \qquad (4)$$

If **r** is regarded as the position vector of the point P relative to O, then x, y, z are the rectangular Cartesian coordinats of P with respect to the axes OX, OY, OZ. For points P_1, P_2 with position vectors r_1, r_2 we have

$$\overrightarrow{P_1P_2} = \mathbf{r}_2 - \mathbf{r}_1 = (x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k},$$

and consequently $\quad P_1P_2{}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2,$

given the square of the distance between two points in terms of their rectangular Cartesian coordinates.

It will frequently be convenient to denote a vector of the form x**i** + y**j** + z**k** simply by (x, y, z). A point P, with this as position vector, will be referred to as the point P(x, y, z). Any unit vector is expressible as (*l*, m, n where *l*, m, n are its direction cosines. for subtraction of vectors we have

$$\mathbf{r}_1 - \mathbf{r}_2 = (x_1, y_1, z_1) - (x_2, y_2, z_2) = (x_1 - x_2, y_1 - y_2, z_1 - z_2),$$

and, for multiplication by a real number, we have m(x, y, z) = (mx, my, mz).

The modulus of the vector (a_1, a_2, a_3) is a $\sqrt{(a_1{}^2 + a_2{}^2 + a_3{}^2)}$, and the unit vector in this direction is $(a_1, a_2, a_3)/a$. Also, the position vector of the point R, which divides the join of $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ in the ratio m : n, is

$$\mathbf{r} = \left(\frac{na_1 + mb_1}{m+n}, \frac{na_2 + mb_2}{m+n}, \frac{na_3 + mb_3}{m+n}\right).$$

Direction Cosines :

Let $\vec{a} = a_1 i + a_2 j + a_3 k$ the angles which this vector makes with the +ve directions OX,OY & OZ are called **DIRECTION ANGLES** & their cosines are called the **DIRECTION COSINES** .

$\cos\alpha = \dfrac{a_1}{|\vec{a}|}, \cos\beta = \dfrac{a_2}{|\vec{a}|}, \cos\gamma = \dfrac{a_3}{|\vec{a}|}$. **Note that, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$**

Ex.5 A man travelling East at 8 miles an hour finds that the wind seems to blow directly from the North. On doubling his speed he finds that it appears to come from NE. Find the velocity of the wind.

Sol. Let **i, j** represent velocities of 8 milies an hour toward E. and N. respectively. Then the original velcotiy of the man is **i**. Let that of the wind be x**i** + y**j**. Then the velocity of the wind relative to the man is

$$(x\mathbf{i} + y\mathbf{j}) - \mathbf{i}.$$

But this is form the N., and is therefore parallel to –**j**. Hence x = 1.

When the man doubles his speed the velocity of the wind relative to him is

$$(x\mathbf{i} + y\mathbf{j}) - 2\mathbf{i}.$$

But this is form NE., and is therefore parallel to – (**i** + **j**). Hence

$$y = x - 2 = -1.$$

Thus the velocity of the wind is **i** – **j**, which is equivalent ot 8 $\sqrt{2}$ miles an hour from NW.

Ex.6 Given $\vec{a} = 3\hat{i} + 2\hat{j} + 4\hat{k}\,;\vec{b} = 2\left(\hat{i} + \hat{k}\right)$ and $\vec{c} = 4\hat{i} + 2\hat{j} + 3\hat{k}$. For what values of '$\alpha$' the equation , $x\vec{a} + y\vec{b} + z\vec{c} = \alpha\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$ has a non trivial solution .

Sol. Equating the components ,

$$3x + 2y + 4z = \alpha x\,;\ 2x + 2z = \alpha y\ \&\ 4x + 2y + 3z = \alpha z$$

for non trivial solution $\begin{vmatrix} 3-\alpha & 2 & 4 \\ 2 & -\alpha & 2 \\ 4 & 2 & 3-\alpha \end{vmatrix} = 0$

Use : $C_1 \to C_1 - C_2$ $\Rightarrow (\alpha + 1)^2 (\alpha - 8) = 0$ Ans. : $\alpha = -1$ or 8

7. TEST OF COLLINEARITY :

Three points A,B,C with position vectors $\vec{a}, \vec{b}, \vec{c}$ respectively are collinear, if & only if there exist scalars x , y , z not all zero simultaneously such that ; $x\vec{a} + y\vec{b} + z\vec{c} = 0$, where x+ y+ z = 0 .

Theorem of Menelaus. If a transversal cuts the side BC, CA, AB of a triangle in the point P, Q, R respectively, the product of the ratios in which P, Q, R divide those sides is equal to –1.

Let Q divide CA in the ratio m : n. Then we can find a number l such that R divides AB in the ratio l : m. Consequently

$$(m + n)\,\mathbf{q} = n\mathbf{c} + m\mathbf{a}, \quad (l + m)\mathbf{r} = m\mathbf{a} + l\mathbf{b}.$$

Eliminate **a** and write the result in the form

$$\frac{n\mathbf{c} - l\mathbf{b}}{n - l} = \frac{(m+n)\mathbf{q} - (l+m)\mathbf{r}}{n - l} = \mathbf{p}.$$

Thus BC and RQ intersect at P, which divides BC in the ratio –n : 1. consequently the product of the three ratios is –1.

Observing that (n – 1) **p** –(m + n) **q** + (l + m) **r** = **0** show conversely that, if the product of the above ratios is –1, then P, Q, R are collinear.

Theorem of Desargues. If the lines joining corresponding vertices of two triangles are concurrent, the points of intersection of corresponding sides are collinear.

Let A, B, C correspond to D, E, F respectively. Then BC and EF are corresponding sides. Given AD, BE, CF intersect at a point H, we have relations of the form

$$l\mathbf{a} + l'\mathbf{b} = m\mathbf{b} + m'\mathbf{e} = n\mathbf{c} + n'\mathbf{f} = \mathbf{h}$$

where $l + l' = m + m' = n + n' = 1.$

Hence $\dfrac{m\mathbf{b} - n\mathbf{c}}{m - n} = \dfrac{m'\mathbf{e} - n'\mathbf{f}}{m' - n'} = \mathbf{p}$

giving the position vector of the point P of intersection of BC and EF. Write down similar expressions for **q** and **r**, the intersections of the other pairs of corresponding sides, and verify that

$$(m - n)\mathbf{p} + (n - l)\mathbf{q} + (l - m)\mathbf{r} = \mathbf{0},$$

in which the sum of the coefficents is zero. Consequently P, Q, R are collinear.

Theorem of Pascal. If A, B, C are points on one of two intersecting straight lines, and A′, B′, C′ are on the other, then the point P, in which BC′ cuts B′C, is colinear with the points Q, R in which CA′ cuts C′A, and AB′ cuts A′B.

Use the notation and result of Ex.1 and write down the corresponding expressions for **p** and **q**, the position vectors of P and Q. Hence, taking H as origin, show that

$$aa'(cc' - bb')\mathbf{p} + bb'(aa' - cc')\mathbf{q} + cc'(bb' - aa')\mathbf{r} = \mathbf{0},$$

a linear relation in which the sum of the coefficients of **p, q, r** is zero. Consequently the poins P, Q, R are collinear.

Test of Coplanarity :

Four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively are coplanar if and only if there exist scalars x, y, z, w not all zero simultaneously such that $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = 0$ where, x + y + z + w = 0 .

8. SCALAR PRODUCT OF TWO VECTORS :

Scalar quantities are of frequent ocurrence, which depend each upon two vector quantities in such a way as to be jointly proportional to their magnitudes and to the cosine of their mutual inclination. An example of such is the work done by a force during a displacement of the body acted upon. We therefore find it convenient to adopt the following.

Definition. The scalar product of two vectors **a** and **b**, whose directions are inclined at an angle θ, is the real number an cos θ, and is written $\mathbf{a} \cdot \mathbf{b} = ab \cos\theta = \mathbf{b} \cdot \mathbf{a}$.

The order of the factors may be reversed without altering the value of the product. Further, b cos θ is the resolute of **b** in the direction of **a**, and a cos θ is the resolute of **a** in the direction of **b**, positive or negative according as θ is acute or obtuse. Hence

The scalar product of two vectors is the product of the modulus of either vector and the resolute of the other in its direction.

If two vectors **a, b** are perpendicular, cos θ = 0, and their scalar product is zero. Hence the condition of perpendicularity of two proper vectors is expressed by $\mathbf{a} \cdot \mathbf{b} = 0$

If the vectors have the same direction, cos θ = 1, and $\mathbf{a} \cdot \mathbf{b} = ab$. If their directions are opposite, cos θ = –1, and $\mathbf{a} \cdot \mathbf{b} = -ab$. The scalar product of any two unit vectors is equal to the cosine of the angle between their directions.

When the factors are equal vectors their scalar prduct **a . a** is called the square of **a**, and is written $\mathbf{a}^2$. Thus $\mathbf{a}^2 = \mathbf{a} \cdot \mathbf{a} = a^2$,

the square of a vector being thus equal to the square of its modulus. The square of any unit vector is unity. In particular, $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1$,

but since these vectors are mutually perpendicular $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$.

These relations will be constantly employed.

If either factor is multiplied by a number, the scalar product is multiplied by theat number. for $(n\mathbf{a}) \cdot \mathbf{b} = nab \cos\theta = \mathbf{a} \cdot (n\mathbf{b})$.

since the scalar product is a number, it may occur as the numerical coefficient of a vector. Thus $(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ is a vector, obtained on multiplying **c** by the number **a . b**. The combination $(\mathbf{a} \cdot \mathbf{b})(\mathbf{c} \cdot \mathbf{d})$ of four vectors is simply the product of the two numbers **a . b** and **c . d**.

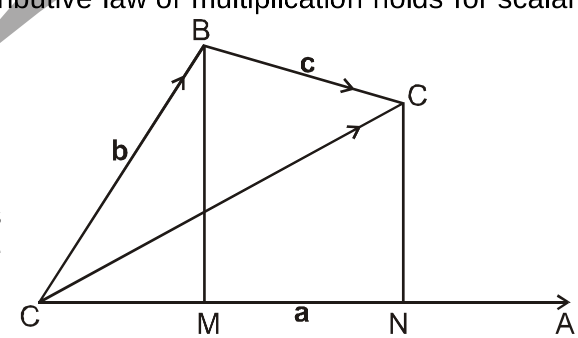
Suppose that a vector **r** is resolved into two component vectors, one in the direction of the unity vector **e**, and the other perpendicular to **e**. The first of these, being the projection of **r** on **e**, is $(\mathbf{r} \cdot \mathbf{e})\mathbf{e}$. The component vector perpendicular to **e** is therefore $\mathbf{r} - (\mathbf{r} \cdot \mathbf{e})\mathbf{e}$.

Similarly the rectangular resolution of Art. 7 may be expressed $\mathbf{r} = (\mathbf{r} \cdot \mathbf{i})\mathbf{i} + (\mathbf{r} \cdot \mathbf{j})\mathbf{j} + (\mathbf{r} \cdot \mathbf{k})\mathbf{k}$.

Distributive Law. It is easy to show that the distributive law of multiplication holds for scalar products; that is, $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$.

The resolute of **b + c** in the direction of a is the sum of the resolutes of **b** and **c** in the same direction. Consequnently on multiplying each of these resolutes by **a**, we have the required result.

Fig. is drawn for the case in which all the resolutes are positive; but the argument holds also when one or more of them are negative.



Using the result of the preceding Art. we may write

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot (\overrightarrow{OC}) = a.\ ON$$

$$= a\,(OM + MN) = a \cdot OM + a \cdot MN = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}.$$

Repeated application of this result shows that the scalar product of two sums of vectors may be expanded as in ordinary algebra. Thus

$$(\mathbf{a} + \mathbf{b} + \ldots).(\mathbf{1} + \mathbf{m} + \ldots) = \mathbf{a} \cdot \mathbf{1} + \mathbf{a} \cdot \mathbf{m} + \ldots$$
$$+ \mathbf{b} \cdot \mathbf{1} + \mathbf{b} \cdot \mathbf{m} + \ldots$$
$$+ \ldots$$

In particular, $(\mathbf{a} + \mathbf{b})^2 = \mathbf{a}^2 + 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b}^2$,

while $(\mathbf{a} \cdot \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{a}^2 - \mathbf{b}^2$.

From the distributive law we may deduce a very useful formula for the scalar product of two vectors in terms of their rectangular components, For, if $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, then

$$\mathbf{a} \cdot \mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$$
$$= a_1b_1 + a_2b_2 + a_3b_3$$

since the vectors **i, j, k** are mutually perpendicular. Thus

The scalar product of two vectors is equal to the sum of the product of their corresponding rectangular components.

In particular, the square of a vector is equal to the sum of squares of its rectangular components. Also since the above scalar product is ab cos θ, the inclination of the vectors is given by

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{(a_1^2 + a_2^2 + a_3^2)}\sqrt{(b_1^2 + b_2^2 + b_3^2)}}$$

Ex.7 Prove cosine formula $c^2 = a^2 + b^2 - 2ab\cos C$ in triangle ABC.

Sol. If **a, b, c** are the vectors $\overrightarrow{BC}, \overrightarrow{CA}, \overrightarrow{AB}$ we have $-c = \mathbf{a} + \mathbf{b}$. On 'squaring' both sides of this equation we find $c^2 = \mathbf{c}^2 = \mathbf{a}^2 + \mathbf{b}^2 + 2\mathbf{a} \cdot \mathbf{b} = a^2 + b^2 - 2ab\cos C$,

Since the inclination of **a** and **b** is $\pi - C$.

Ex.8 Show that, if **a, b, c** are non-coplanar vectors, and $\mathbf{p} \cdot \mathbf{a} = \mathbf{p} \cdot \mathbf{b} = \mathbf{p} \cdot \mathbf{c} = 0$, then **p** is the zero vector.

Sol. Since **p** is perpendicular to both **a** and **b** it is normal to the plane of **a** and **b**. Then, since $\mathbf{p} \cdot \mathbf{c} = 0$, **c** must lie in the plane of **a** and **b**. But this is contrary to the data. Hence **p** must be zero.

Ex.9 Show that the perpendiculars from the vertices of a triangle to the opposite sides are concurrent.

Sol. Let **a, b, c** be the position vectors of A, B, C and **h** that of the intersection, H, of the perpendiculars from B and C. then $(\mathbf{h} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{c}) = 0$, $(\mathbf{h} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) = 0$.

Addition of these shows that $(\mathbf{h} - \mathbf{a}) \cdot (\mathbf{a} - \mathbf{c}) = 0$, so that AH is perpendicular to BC, and the theorem is proved.

Ex.10 Show that the perpendicular bisectors of the sides of a triangle are concurrent.

Sol. Let K be the intersection of the perpendicular bisectors of AB and AB. Then, with the notation of previous Ex, $\left(\mathbf{k} - \frac{\mathbf{a} + \mathbf{b}}{2}\right) \cdot (\mathbf{a} - \mathbf{b}) = 0$, $\left(\mathbf{k} - \frac{\mathbf{a} + \mathbf{c}}{2}\right) \cdot (\mathbf{c} - \mathbf{a}) = 0$.

Addition of these shows that $\left(\mathbf{k} - \frac{\mathbf{c} + \mathbf{b}}{2}\right) \cdot (\mathbf{c} - \mathbf{b}) = 0$, so that K is also on the perpendicular bisector of BC.

Ex.11 In a tetrahedron, if two pairs of opposite edges are perpendicular, the third pair are also perpendicular to each other; and the sum of the squares on two opposite edges is the same for each pair.

Sol. We have $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, $\overrightarrow{AC} = \mathbf{c} - \mathbf{a}$ and $\overrightarrow{CB} = \mathbf{b} - \mathbf{c}$. Hence if BD is perpendicular to CA, $\mathbf{b} \cdot (\mathbf{c} - \mathbf{a}) = 0$,

that is $\mathbf{b} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b}$.

And similarly, if DA is perpendicular to BC, $\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0$.

that is $\mathbf{a} \cdot \mathbf{b} = \mathbf{c} \cdot \mathbf{a}$. (i)

Thus $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a}$,

whence $\mathbf{c} \cdot (\mathbf{a} - \mathbf{b}) = 0$,

showing that DC is perpendicular to BA.

Further, the sum of the squares on BD and CA is

$$\mathbf{b}^2 + (\mathbf{c} - \mathbf{a})^2 = \mathbf{b}^2 + \mathbf{c}^2 + \mathbf{a}^2 - 2\mathbf{a} \cdot \mathbf{c},$$

and, in virtue of (i), this is the same as in the other two cases.

Note :

(i) $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta \ (0 \le \theta \le \pi)$,

note that if θ is acute then $\vec{a}.\vec{b} > 0$ & if θ is obtuse then $\vec{a}.\vec{b} < 0$

(ii) $\vec{a}.\vec{a} = |\vec{a}|^2 = \vec{a}^2$, $\vec{a}.\vec{b} = \vec{b}.\vec{a}$ (commutative)

(iii) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$ (distributive)

(iv) $\vec{a}.\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$ $\quad (\vec{a} \ne 0 \quad \vec{b} \ne 0)$

(v) i.i = j.j = k.k = 1 ; i.j = j.k = k.i = 0

(vi) projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

(vii) the angle ϕ between $\vec{a}$ & $\vec{b}$ is given by $\cos\phi = \dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$ $\quad 0 \le \phi \le \pi$

(viii) if $\vec{a} = a_1 i + a_2 j + a_3 k$ & $\vec{b} = b_1 i + b_2 j + b_3 k$ then $\vec{a}.\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, $|\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$

(ix) Maximum value of $\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|$

(x) Minimum values of $\vec{a}.\vec{b} = \vec{a}.\vec{b} = -|\vec{a}|\,|\vec{b}|$

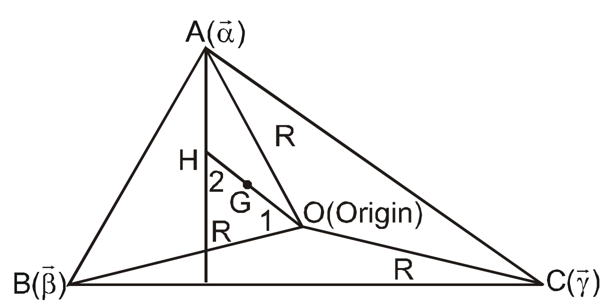
(xi) Any vector $\vec{a}$ can be written as , $\vec{a} = (\vec{a} \cdot \hat{i})\,\hat{i} + (\vec{a} \cdot \hat{j})\,\hat{j} + (\vec{a} \cdot \hat{k})\,\hat{k}$.

(xii) A vector in the direction of the bisector of the angle between the two vectors $\vec{a}\,\&\,\vec{b}$ is $\dfrac{\vec{a}}{|\vec{a}|} + \dfrac{\vec{b}}{|\vec{b}|}$. Hence bisector of the angle between the two vectors $\vec{a}\,\&\,\vec{b}$ is $\lambda\left(\hat{a} + \hat{b}\right)$, where $\lambda \in R^+$. Bisector of the exterior angle between $\vec{a}\,\&\,\vec{b}$ is $\lambda\left(\hat{a} - \hat{b}\right)$, $\lambda \in R^+$.

Ex.12 In a ΔABC if 'O' is the circumcentre, H is the orthocentre and R is the radius of the circle circumscribing the triangle ABC, then prove that ;

(i) $\vec{OH} = \vec{OA} + \vec{OB} + \vec{OC}$ (ii) $\left|\vec{OH}\right|^2 = 9R^2 - (a^2 + b^2 + c^2)$

(iii) $\left|\vec{AH}\right| = 2R\,|\cos A|$



Sol.

(i) position vector of G are $\dfrac{\vec{\alpha} + \vec{\beta} + \vec{\gamma}}{3} = \vec{OG}$

$\Rightarrow \vec{OH} = 3\,\vec{OG} = \vec{\alpha} + \vec{\beta} + \vec{\gamma}$

$= \vec{OA} + \vec{OB} + \vec{OC}$

(ii) $\left|\vec{OH}\right|^2 = (\vec{\alpha} + \vec{\beta} + \vec{\gamma})^2 = 3R^2 + 2R^2\,[\Sigma \cos 2A]$ $= 3R^2 + 2R^2\,[3 - 2(\Sigma \sin^2 A)]$

$= 9R^2 - 4R^2\left(\dfrac{a^2}{4R^2} + \dfrac{b^2}{4R^2} + \dfrac{c^2}{4R^2}\right)$ $= 9R^2 - (a^2 + b^2 + c^2)$

(iii) $\overrightarrow{AH}$ = position vector of $\vec{H}$ – position vector of $\vec{A}$ $= \vec{\alpha} + \vec{\beta} + \vec{\gamma} - \vec{\alpha} = \vec{\beta} + \vec{\gamma}$

$\left|\overrightarrow{AH}\right|^2 = (\vec{\beta} + \vec{\gamma})^2 = 2R^2 + 2R^2 \cos 2A = 4R^2 \cos^2 A$

Ex.13 Let $\vec{u}$ be a vector on rectangular coordinate system with sloping angle 60°. Suppose that $\left|\vec{u} - \hat{i}\right|$ is geometric mean of $|\vec{u}|$ and $\left|\vec{u} - 2\hat{i}\right|$ where $\hat{i}$ is the unit vector along x-axis then $|\vec{u}|$ has the value equal to $\sqrt{a} - \sqrt{b}$ where a, b $\in$ N, find the value $(a + b)^3 + (a - b)^3$.

Sol. Let $\vec{u} = x\hat{i} + \sqrt{3}x\hat{j}$; $|\vec{u}| = 2x,\ x > 0$

now $|\vec{u}|\left|\vec{u} - 2\hat{i}\right| = \left|\vec{u} - \hat{i}\right|^2$

$2\,|x|\sqrt{(x-2)^2 + 3x^2} = [(x-1)^2 + 3x^2]$

$2\,|x|\sqrt{4x^2 - 4x + 4} = 4x^2 - 2x + 1$

$4\,|x|\sqrt{x^2 - x + 1} = 4x^2 - 2x + 1$

square $16x^2(x^2 - x + 1) = 16x^4 + 4x^2 + 1 - 16x^3 - 4x + 8x^2$

$16x^2 = 12x^2 + 1 - 4x$

$4x^2 + 4x - 1 = 0$

$x = \dfrac{-4 \pm \sqrt{16+16}}{8} = \dfrac{-4 \pm 4\sqrt{2}}{8} = \dfrac{-1 \pm \sqrt{2}}{2}$ or $\dfrac{-(1+\sqrt{2})}{2}$

$2x = \sqrt{2} - 1$ or $-(\sqrt{2} + 1) \rightarrow$ rejected

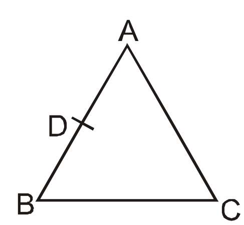
hence $|\vec{u}| = \sqrt{2} - 1 = \sqrt{2} - \sqrt{1}$ $\Rightarrow$ a = 2; b = 1

$(a + b)^3 + (a - b)^3 = 27 + 1 = 28$ Ans.

Ex.14 In a triangle ABC, AB = AC and O is its circumcentre. Also D is themidpoint of AB and E is the centroid of triangle ACD. Using vector method, prove that CD is perpendicular to OE.

Sol. Take A as the initial point and the p.v. of B, C and O as $\vec{b}, \vec{c}$ and $\vec{r}$ respectively.

The p.v. of D is $\dfrac{\vec{b}}{2}$ and that of E $\dfrac{2\vec{c} + \vec{r}}{6}$. Since O is the circumcentre of



ΔABC, $|\vec{r}| = |\vec{r} - \vec{c}| = |\vec{r} - \vec{b}|$

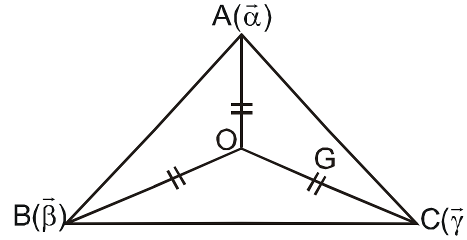
$\Rightarrow \quad \vec{r}\,.\,\vec{b} = \vec{r}\,.\,\vec{c} = \dfrac{|\vec{b}|^2}{2} = \dfrac{|\vec{c}|^2}{2}$.

Hence $\overrightarrow{CD}\,.\,\overrightarrow{OE} = \left(\vec{c} - \dfrac{\vec{b}}{2}\right)\left(\dfrac{2\vec{c} + \vec{b}}{6} - \vec{r}\right) = \dfrac{4|\vec{c}|^2 - |\vec{b}|^2}{12} - \vec{r}\,.\,\vec{c} + \dfrac{1}{2}\vec{r}\,.\,\vec{b} = \dfrac{|\vec{b}|^2}{4} - \dfrac{|\vec{b}|^2}{2} + \dfrac{|\vec{b}|^2}{4} = 0$

$\Rightarrow$ CD is perpendicular to OE.

Ex.15 Prove using vectors that the distance of the circumcenter of the Δ ABC from the centroid is $\sqrt{R^2 - \dfrac{1}{9}(a^2 + b^2 + c^2)}$ where R is the circumradius.

Sol. circumcenter is 'O' $\left|\overrightarrow{OG}\right|^2 = \dfrac{1}{9}\left|\vec{\alpha} + \vec{\beta} + \vec{\gamma}\right|^2$



$$= \frac{1}{9}\left(\left|\vec{\alpha}\right|^2 + \left|\vec{\beta}\right|^2 + \left|\vec{\gamma}\right|^2 + 2\,\vec{\alpha}\cdot\vec{\beta} + 2\,\vec{\beta}\cdot\vec{\gamma} + 2\,\vec{\gamma}\cdot\vec{\alpha}\right)$$

$$= \frac{1}{9}\left[3R^2 + 2R^2(\cos 2A + \cos 2B + \cos 2C)\right]$$

$$= \frac{1}{9}\left[3R^2 + 2R^2(3 - 2\sin^2 A)\right] \qquad = \frac{1}{9}\left[9R^2 - 4R^2\left(\frac{a^2}{4R^2} + \frac{b^2}{4R^2} + \frac{c^2}{4R^2}\right)\right]$$

Ex.16 Let ABC be a triangle with AB = AC. If D is the mid point of BC, E the foot of the perpendicular drawn from D to AC and F the mid point of DE, use vector methods to prove that AF is perpendicular to BE.

Sol. Let D be the initial point. Let the position vectors of

A, B, C be $\vec{a}, \vec{b}, -\vec{b}$ respectively.

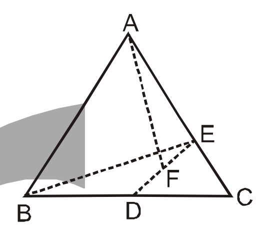
It is given that AB = AC $\Rightarrow$ AD is perpendicular to

BC $\Rightarrow$ $\vec{a}.\vec{b} = 0$,

If E divides AC in the ratio 1 : λ, then p.v. of E is $\dfrac{\lambda\vec{a} - \vec{b}}{1+\lambda}$.

Since DE is perpendicular to AC, $\overrightarrow{DE}\,.\,\overrightarrow{AC} = 0$

$\Rightarrow \quad \left(\dfrac{\lambda\vec{a} - \vec{b}}{1+\lambda}\right).(\vec{a} + \vec{b}) = 0 \quad \Rightarrow \quad \lambda|\vec{a}|^2 = |\vec{b}|^2 = 0 \Rightarrow \lambda = \dfrac{|\vec{b}|^2}{|\vec{a}|^2}$, so that



p.v. of E is $\dfrac{|\vec{b}|^2\,\vec{a} - |\vec{a}|^2\,\vec{b}}{|\vec{a}|^2 + |\vec{b}|^2} \quad \Rightarrow \quad$ p.v. of F is $\dfrac{|\vec{b}|^2\,\vec{a} - |\vec{a}|^2\,\vec{b}}{2(|\vec{a}|^2 + |\vec{b}|^2)}$

Vector $\overrightarrow{AF} = \vec{a}\,\dfrac{|\vec{b}|^2\,\vec{a} - |\vec{a}|^2\,\vec{b}}{2(|\vec{a}|^2 + |\vec{b}|^2)} = \dfrac{(2|\vec{a}|^2 + |\vec{b}|^2)\vec{a} + |\vec{a}|^2\,\vec{b}}{2(|\vec{a}|^2 + |\vec{b}|^2)}$

Vector $\overrightarrow{BE} = \vec{b}\,\dfrac{|\vec{b}|^2\,\vec{a} - |\vec{a}|^2\,\vec{b}}{|\vec{a}|^2 + |\vec{b}|^2} = \dfrac{(2|\vec{a}|^2 + |\vec{b}|^2)\vec{b} + |\vec{b}|^2\,\vec{a}}{|\vec{a}|^2 + |\vec{b}|^2}$

Hence $\overrightarrow{AF}\,.\,\overrightarrow{BE} = \dfrac{|\vec{a}|^2\,|\vec{b}|^2\,(-2|\vec{a}|^2 - |\vec{b}|^2 + 2|\vec{a}|^2 + |\vec{b}|^2)}{2(|\vec{a}|^2 + |\vec{b}|^2)} = 0$

$\Rightarrow$ AF is perpendicular to BE.

Ex.17 A_1A_2......A_n be an n sided regular polygon circumscribed over a circle of radius r. If P be any point on circle. Using vectors prove that $PA_1 + PA_2 + ... + PA_n \le \sqrt{3nr^2\left(1 + \sec^2\dfrac{\pi}{n}\right)}$.

Sol. Let centre be point of reference O.

$\Rightarrow \quad \overrightarrow{OA_1} + \overrightarrow{OA} + \overrightarrow{OA_3} + ... + \overrightarrow{OA_n} = \vec{O}$ ...(1)

$\sum(\overrightarrow{PA_i})^2 = \sum(\overrightarrow{OP} - \overrightarrow{OA_i})^2$

$= \sum(\overrightarrow{OP})^2 + \sum(\overrightarrow{OA_i}))^2 - 2\sum\overrightarrow{OP}\,.\,\overrightarrow{OA_i}$

$= nr^2 + nr^2\sec^2\dfrac{\pi}{n}$

and $|\overrightarrow{OA_i}| = r \sec \frac{\pi}{n}$

Now, $\left|\sum(\overrightarrow{PA_i})\right|^2 = \sum\left|(\overrightarrow{PA_i})\right|^2 + 2\sum|\overrightarrow{PA_1}|.|\overrightarrow{PA_2}|$

Also, $\sum(\overrightarrow{PA_i})^2 \geq \sum|\overrightarrow{PA_1}|.|\overrightarrow{PA_2}|$

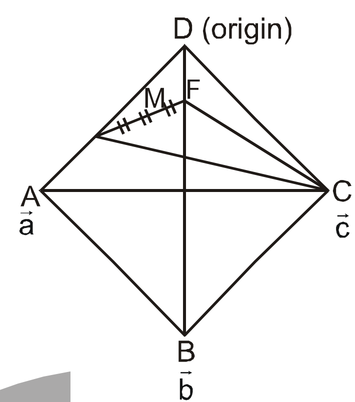
$\Rightarrow \quad 3\sum(\overrightarrow{PA_i})^2 \geq \left(\sum|\overrightarrow{PA_i}|\right)^2$

$3nr^2\left(1+\sec^2\frac{\pi}{n}\right) \geq \left(\sum|\overrightarrow{PA_i}|\right)^2$

$\Rightarrow \quad PA_1 + PA_2 + ... + PA_n \leq \sqrt{3nr^2\left(1+\sec^2\frac{\pi}{n}\right)}$

Ex.18 The length of the edge of the regular tetrahedron $D-ABC$ is 'a'. Point E and F are taken on the edges AD and BD respectively such that E divides $\overrightarrow{DA}$ and F divides $\overrightarrow{BD}$ in the ratio 2 : 1 each. Then find the area of triangle CEF .

Sol. We have $|\vec{a}| = |\vec{b}| = |\vec{c}| = |\vec{b}-\vec{a}| = |\vec{b}-\vec{c}| = |\vec{c}-\vec{a}| = a$



and $\hat{\vec{a}\ \vec{b}} = \hat{\vec{b}\ \vec{c}} = \hat{\vec{c}\ \vec{a}} = \hat{\overrightarrow{AB}\ \overrightarrow{AC}} = \hat{\overrightarrow{CA}\ \overrightarrow{CB}} = \hat{\overrightarrow{BA}\ \overrightarrow{BC}} = \frac{\pi}{3}$

$(\overrightarrow{EF})^2 = \frac{a^2}{3} \Rightarrow \overrightarrow{EF} = \frac{a}{3}$

$|\overrightarrow{CF}| = |\overrightarrow{CE}| = \frac{\sqrt{7}\,a}{3}\ |\overrightarrow{CM}| = \frac{5a}{6}$ where M is the middle point of EF .

Area (DCEF) $= \frac{1}{2}\,|\overrightarrow{EF}|\,|\overrightarrow{CM}| = \frac{1}{2}.\frac{a}{\sqrt{3}}.\frac{5a}{6} = \frac{5a^2}{12\sqrt{3}}$

Ex.19 Given three points on the xy plane on O(0, 0), A(1, 0) and B(–1, 0). Point P is moving on the plane satisfying the condition $(\overrightarrow{PA}\cdot\overrightarrow{PB}) + 3(\overrightarrow{OA}\cdot\overrightarrow{OB}) = 0$

If the maximum and minimum values of $|\overrightarrow{PA}||\overrightarrow{PB}|$ are M and *m* respectively then find the value of $M^2 + m^2$. [Ans. 34]

Sol. Let P be (x, y)

$\overrightarrow{PA} = (1-x)\hat{i} - y\hat{j}; \quad \overrightarrow{PB} = (-1-x)\hat{i} - y\hat{j}$

$\therefore \quad (\overrightarrow{PA}\cdot\overrightarrow{PB}) = \left((x-1)\hat{i}+y\hat{j}\right)\cdot\left((x+1)\hat{i}+y\hat{j}\right) = (x^2-1) + y^2$

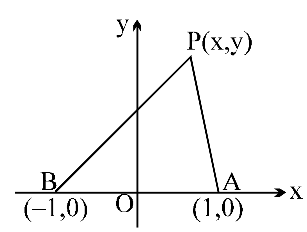
also $3(\overrightarrow{OA}\cdot\overrightarrow{OB}) = 3\hat{i}\cdot(-\hat{i}) = -3$

hence $(\overrightarrow{PA}\cdot\overrightarrow{PB}) + 3(\overrightarrow{OA}\cdot\overrightarrow{OB}) = 0$

$x^2 - 1 + y^2 - 3 = 0 = 0$

$x^2 + y^2 = 4$ ....(1)

which gives the locus of P i.e. P move on a circle with centre (0, 0) and radius 2.



now $\left|\overrightarrow{PA}\right|^2 = (x-1)^2 + y^2$; $\left|\overrightarrow{PB}\right|^2 = (x+1)^2 + y^2$

$\therefore$ $\left|\overrightarrow{PA}\right|^2 \left|\overrightarrow{PB}\right|^2 = (x^2 + y^2 - 2x + 1)(x^2 + y^2 + 2x + 1)$

$= (5 - 2x)(5 + 2x)$ [using $x^2 + y^2 = 4$]

$\therefore$ $\left|\overrightarrow{PA}\right|^2 \left|\overrightarrow{PB}\right|^2 = 25 - 4x^2$ subject to $x^2 + y^2 = 4$

$\left.\left|\overrightarrow{PA}\right|^2 \left|\overrightarrow{PB}\right|^2\right|_{\min.} = 25 - 16 = 9$; (when x = 2 or – 2)

and $\left.\left|\overrightarrow{PA}\right|^2 \left|\overrightarrow{PB}\right|^2\right|_{\max.} = 25 - 0 = 25$ (when x = 0)

$3 \le \left|\overrightarrow{PA}\right| \left|\overrightarrow{PB}\right| \le 5$

hence M = 5 and m = 3 $\Rightarrow$ $M^2 + m^2 = 34$ Ans.

9. VECTOR PRODUCT OF TWO VECTORS :

Vector quantities are of frequent occurrence, which depend each upon two other vector quantities in such a way as to be jointly proportional to their magnitudes and to the sine of their mutual inclination, and to have a direction perpendicular to each of them. We are therefore led to adopt the following

Definition. The vector product of two vectors **a** and **b**, whose directions are inclined at an angle θ, is the vector whose modulus si ab sin θ, and whose direction is perpendiculat to both **a** and **b**, being positive relative to a rotation from **a** to **b**.

We write it **a** × **b**, so that **a** × **b** = ab sin θ **n**,

where **n** is a unit vector perpendicular to the plane of **a**, **b**, having the same direction as the translation of a right-handed screw due to a rotation from **a** to **b**. From this it follows that **b** × **a** has the opposite direction to **a** × **b**, but the same length, so that **a** × **b** = – **b** × **a**.

The order of the factors in vector product is not commutative; for a reversal of the order alters the sign of the product.

Consider the parallelogram OAPB whose sides OA, OB have the lengths and directions of **a** and **b** respectively. The area of the figure is ab sin θ, and the vector area OAPB, whose boundary is described in this sense, is represented by ab sin θ **n** = **a** × **b**. This simple gemoterical relation will be found useful. The vector area OBPA is of course represented by **b** × **a**.

For two parallel vectors sin θ is zero, adn their vector product vanishes. The relation **a** × **b** = **0** is thus the condition of parallelism of two proper vectors. In particular, **r** × **r** = **0** is true for all vectors.

If, however, **a** and **b** are perpendicular, **a** × **b** is a vector whose modulus is ab, and whose direction is such that **a**, **b**, **a** × **b** form a right-handed system of mutually perpendicular vectors.

If **a, b** are both nit vectors the modulus of **a** × **b** is the sine of their angle of inclination. For the particular unit vectors, **i, j, k**

we have **i** × **i** = **j** × **j** = **k** × **k** = **0**,

while **i** × **j** = **k** = –**j** × **i**,

j × **k** = **i** = –**k** × **j**,

k × **i** = **j** = –**i** × **k**.

These relations will be constantly employed.

If either factor is multiplied by a number, their product is multiplied by that number. For

(m**a**) × **b** = mab sin θ **n** = **a** × (m**b**).

Distributive Law. We shall now show that the distributive law holds for vector products also; that is

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}. \qquad (1)$$

From the distributive law may be deduced a very useful formula for the vector product **a** × **b** in terms of rectangular components of the vectors. For, with the usual notation,

$$\mathbf{a} \times \mathbf{b}\ (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$$
$$= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}, \qquad (2)$$

in virtue of the relation provedc in the preceding Art. We may write this in the determinantal form

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \end{vmatrix} \qquad (3)$$

This vector has modulus ab sin θ. Hence, on squaring both members of the above equation and dividing by a^2b^2, we find for the sine of the angle between two vectors **a** and **b**,

$$\sin^2\theta = \frac{(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2}{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)}.$$

If *l*, m, n and *l*′, m′, n′ are the direction cosines of **a** and **b** respectively, this is equivalent to

$$\sin^2\theta = (mn' - m'n)^2 + (nl' - n'l)^2 + (lm' - l'm)^2.$$

It is worth noticing that if **b** = **c** + n**a**, where n is any real number, then

$$\mathbf{a} \times \mathbf{b} = \mathbf{a} \times (\mathbf{c} + n\mathbf{a}) = \mathbf{a} \times \mathbf{c}.$$

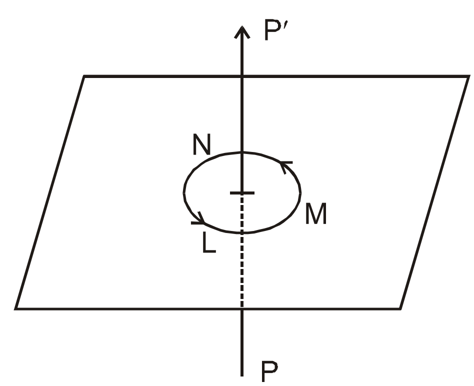
Conversely, if **a** × **b** = **a** × **c** it does not follow that **b** = **c**, but that **b** differs from **c** by some vector parallel to **a**, whic may or may not be zero.

Note :

(i) If $\vec{a}\,\&\,\vec{b}$ are two vectors & θ is the angle between them then $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\ \vec{n}$, where $\vec{n}$ is the unit vector perpendicular to both $\vec{a}\,\&\,\vec{b}$ such that $\vec{a}$, $\vec{b}$ & $\vec{n}$ forms a right handed screw system .

Vector area. Consider the type of vector quantity whose magnitude is an area. Such a quantity is associated with each plane figure, the magnitude being the area of the figure, and the direction that of the normal to the plane of the figure. This vector area therefore specifies both the area and the orientation of the plane figure. But as the direction might be either of two oposite directions along the normal, some convection is neccessary. The area clearly has no sign in itself, and can be regarded as positive or negative only with reference to the direction in which the boundary of the figure is described, or the side of the plane from which it is viewed.

Consider the area of the figure bounded by the closed curve LMN, which is regarded s being traced out in the direction of the arrows. the normal vector $\overrightarrow{PP'}$ bears to this direction of rotation the same relation as the translation to the direction of rotation of a right-handed screw. The area LMN is regarded as positive relative to the direction of $\overrightarrow{PP'}$.



With this convention a vector area may be represented by a vector normal to the plane of the figure, in the direction relative to which it is positive, and with modulus equal to the measure of the area. The sum of two vector areas representd by **a** and **b** is defined to be the vector area

represented by **a** + **b**.

(ii) Geometrically $|\vec{a} \times \vec{b}|$ = area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ & $\vec{b}$.

(iii) $\vec{a} \times \vec{b} = 0 \Leftrightarrow \vec{a}$ & $\vec{b}$ are parallel (collinear) $(\vec{a} \neq 0\ ,\ \vec{b} \neq 0)$ i.e. $\vec{a} = K\vec{b}$, where K is a scalar.

(iv) $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ (not commutative)

(v) $(m\vec{a}) \times \vec{b} = \vec{a} \times (m\vec{b}) = m(\vec{a} \times \vec{b})$ where m is a scalar .

(vi) $\vec{a} \times (\vec{b}+\vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$ (distributive)

(vii) i x i = j x j = k x k = 0

(viii) i x j = k, j x k = i, k x i = j

(ix) If $\vec{a} = a_1 i + a_2 j + a_3 k$ & $\vec{b} = b_1 i + b_2 j + b_3 k$ then $\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

(x) Unit vector perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is $\hat{n} = \pm \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

(xi) A vector of magnitude 'r' & perpendicular to the palne of $\vec{a}$ & $\vec{b}$ is $\pm \dfrac{r(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$

(xi) If θ is the angle between $\vec{a}$ & $\vec{b}$ then $\sin\theta = \dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$

(xii) If $\vec{a}, \vec{b}$ & $\vec{c}$ are the pv's of 3 points A, B & C then the vector area of triangle ABC = $\dfrac{1}{2}\left[\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}\right]$. The points A, B & C are collinear if $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = 0$

(xiii) Area of any quadrilateral whose diagonal vectors are $\vec{d}_1$ & $\vec{d}_2$ is given by $\dfrac{1}{2}|\vec{d}_1 \times \vec{d}_2|$

(xiv) Lagranges Identity : for any two vectors $\vec{a}$ & $\vec{b}$; $(\vec{a} \times \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a}.\vec{b})^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}$

Partly rectangular resolution of vector. Let **u, v** be any two non-parallel unit vectors inclined at an angle θ. Then we may resolve any vector **a** into component vectors, in the direction of **u, v** and a third directin perpendicular to their plane. We may write

$$\mathbf{u} \times \mathbf{v} = \mathbf{n} \sin\theta \qquad (1)$$

where **n** is a unit vector perpendicular to the palne of **u, v** and directed from the paper to the reader. Other unit vectors **l, m** may then be defined by

$$\mathbf{v} \times \mathbf{n} = \mathbf{l}, \qquad \mathbf{n} \times \mathbf{u} = \mathbf{m}. \qquad (2)$$

These vectors are coplanar with **u** and **v**, **l** being perpendicular to **v**, and **m** to **u**. Also **l** and **m** are inclined at an angle $\pi - \theta$;and corresponding to (1) and (2) we have the eqations

$$\mathbf{l} \times \mathbf{m} = \mathbf{n} \sin\theta, \quad \mathbf{m} \times \mathbf{n} = \mathbf{u}, \quad \mathbf{n} \times \mathbf{l} = \mathbf{v} \qquad (3)$$

Since **u, v, n** are not coplanar we maywrite

$$\mathbf{a} = \lambda\mathbf{u} + \mu\mathbf{v} + \nu\mathbf{n} \qquad (4)$$

Forming the scalar product of each member with **l** we obtain

$$\mathbf{a} \,.\, \mathbf{l} = \lambda\mathbf{u} \,.\, \mathbf{l} = \lambda \sin\theta,$$

so that $\lambda = \mathbf{a} \,.\, \mathbf{l}/\sin\theta$. Similarly, on forming the scalar product of each member of (4) with **m** and **n** in turn, we find $\mu = \mathbf{a} \,.\, \mathbf{m}/\sin\theta$, $\quad \nu = \mathbf{a} \,.\, \mathbf{n}$.

Hence the required resolution

$$\mathbf{a} = (\mathbf{a} \,.\, \mathbf{n})\,\mathbf{n} + \frac{\mathbf{a.l}}{\sin\theta}\mathbf{u} + \frac{\mathbf{a.m}}{\sin\theta}\mathbf{v} \qquad (5)$$

And, since **n, u, v** are obtianed from **l, m** in the same way as **n, l, m** from **u, v** we have the coresponding resolution

$$\mathbf{a} = (\mathbf{a} \,.\, \mathbf{n})\mathbf{n} + \frac{\mathbf{a.u}}{\sin\theta}\mathbf{l} + \frac{\mathbf{a.v}}{\sin\theta}\mathbf{m} \qquad (6)$$

From the vector product of each member of (6) with $\mathbf{u} \times \mathbf{v}$, which by (1) is equal to $\mathbf{n}\sin\theta$. Then

$$\mathbf{a} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{a} \times (\mathbf{n}\sin\theta) = (\mathbf{a} \,.\, \mathbf{u})\mathbf{l} \times \mathbf{n} + (\mathbf{a} \,.\, \mathbf{v})\,\mathbf{m} \times \mathbf{n}$$

$$= (\mathbf{a} \,.\, \mathbf{v})\mathbf{u} - (\mathbf{a} \,.\, \mathbf{u})\mathbf{v} \qquad (7)$$

Ex.20 Given a parallelogram ABCD with area 12 sq. units . A straight line is drawn through the mid point M of the side BC and the vertex A which cuts the diagonal BD at a point ' O ' . Use vectors to determine the area of the quadrilateral OMCD .

Sol. $\vec{r}_1 = \lambda\, \dfrac{2\vec{b} + \vec{d}}{2}$

$\vec{r}_2 = \vec{b} + \mu\, (\vec{b} - \vec{d})$

$\vec{r}_1 = \vec{r}_2$ (given)

$\lambda = \dfrac{2}{3}$ & $\mu = -\dfrac{1}{3}$

$\therefore$ p.v. of ' O ' are $\dfrac{2\vec{b} + \vec{d}}{3}$

Given $|\vec{b} \times \vec{d}| = 12$

Area of quad. OMCD $= \dfrac{1}{2}\,|\vec{d}_1 \times \vec{d}_2|$

$$= \frac{1}{2}\left|\left(\vec{d} - \frac{2\vec{b} + \vec{d}}{2}\right) \times \left(\frac{2\vec{b} + \vec{d}}{3} - (\vec{b} + \vec{d})\right)\right| \qquad = \frac{1}{12}\,|5\vec{b} \times \vec{d}| = 5 \text{ sq. units}$$

Ex.21 $\hat{u}$ and $\hat{v}$ are two non-colinear unit vectors such that $\left|\dfrac{\hat{u} + \hat{v}}{2} + \hat{u} \times \hat{v}\right| = 1$. Prove that $|\hat{u} \times \hat{v}| = \left|\dfrac{\hat{u} + \hat{v}}{2}\right|$.

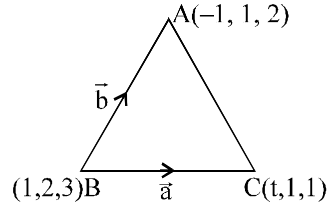
Sol. Given that $\left|\dfrac{\hat{u} + \hat{v}}{2} + \hat{u} \times \hat{v}\right| = 1 \Rightarrow \quad \left|\dfrac{\hat{u} + \hat{v}}{2} + \hat{u} \times \hat{v}\right|^2 = 1 \quad \Rightarrow \quad \dfrac{2 + 2\cos\theta}{4} + \sin^2\theta = 1$

$\Rightarrow \quad \cos^2\dfrac{\theta}{2} = \cos^2\theta \Rightarrow \theta - n\pi \pm \dfrac{\theta}{2} \Rightarrow \theta = \dfrac{2\pi}{3}. \quad \Rightarrow \quad |\hat{u} \times \hat{v}| = \sin\dfrac{2\pi}{3} = \sin\dfrac{\pi}{3} = \left|\dfrac{\hat{u} - \hat{v}}{2}\right|.$

Ex.22 Let A_m be the minimum area of the triangle whose vertices are A(–1, 1, 2); B(1, 2, 3) and C(t, 1, 1) where t is a real number. Compute the value of $(1338\sqrt{3})(A_{min})$.

Sol. $A = \dfrac{1}{2}|\vec{a} \times \vec{b}|$ and $|\vec{a} \times \vec{b}|^2 = \vec{a}^2\,\vec{b}^2 - (\vec{a}^2 \cdot \vec{b}^2)^2$

$\vec{a} = (t-1)\hat{i} - \hat{j} - 2\hat{k}$; $\vec{b} = 2\hat{i} + \hat{j} + \hat{k}$



$|\vec{a}|^2 = (t-1)^2 + 1 + 4$; $|\vec{b}|^2 = 4 + 1 + 1 = 6$

$\vec{a}\cdot\vec{b} = 2(t-1) - 1 - 2 = 2t - 5$

$|\vec{a}\times\vec{b}|^2 = 6[t^2 - 2t + 6] - (4t^2 + 25 - 20t)$

$|\vec{a}\times\vec{b}|^2 = 2t^2 + 8t + 11$ which is minimum at $t = -2$

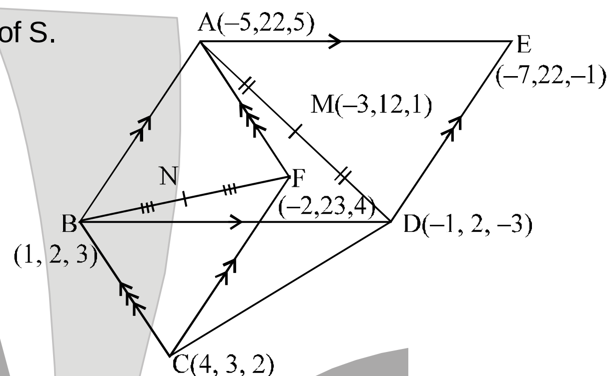
$|\vec{a}\times\vec{b}|^2_{min} = 8 - 16 + 11 = 3$

$|\vec{a}\times\vec{b}|_{min} = \sqrt{3}$

$\therefore \quad \dfrac{|\vec{a}\times\vec{b}|_{min}}{2} = A_{min} = \dfrac{\sqrt{3}}{2}$

hence $\left(1338\sqrt{3}\right)\cdot\dfrac{\sqrt{3}}{2} = 669 \times 3 = 2007$ Ans.

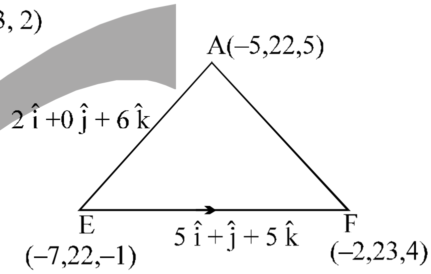
Ex.23 ABCD is a tetrahedron with pv's of its angular points as A(–5, 22, 5); B(1, 2, 3); C(4, 3, 2) and D(–1, 2, – 3). If the area of the triangle AEF where the quadrilaterals ABDE and ABCF are parallelograms is $\sqrt{S}$ then find the value of S.



Sol. pv of M = $\dfrac{\vec{a}+\vec{d}}{2} = -3\hat{i} + 12\hat{j} + \hat{k}$

|||ly pv of N = $\dfrac{\vec{a}+\vec{c}}{2} = -\dfrac{1}{2}\hat{i} + \dfrac{25}{2}\hat{j} + \dfrac{7}{2}\hat{k}$

Now the ΔAEF is as shown

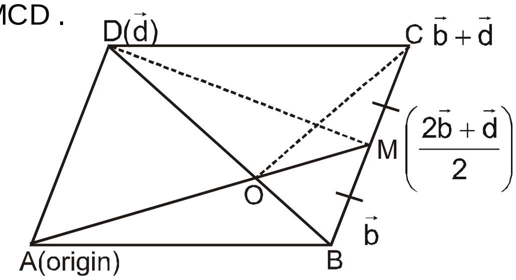


$$\vec{S} = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 6 \\ 5 & 1 & 5 \end{vmatrix}$$

$\left|\vec{S}\right| = \left|-3\hat{i} + 10\hat{j} + \hat{k}\right| = \sqrt{110}$

$\therefore$ S = 110 Ans.]

Ex.24 Given a parallelogram ABCD with area 12 sq. units . A straight line is drawn through the mid point M of the side BC and the vertex A which cuts the diagonal BD at a point ' O ' . Use vectors to determine the area of the quadrilateral OMCD .



Sol. $\vec{r}_1 = \lambda\,\dfrac{2\vec{b}+\vec{d}}{2}$

$\vec{r}_2 = \vec{b} + \mu\left(\vec{b} - \vec{d}\right)$

$\vec{r}_1 = \vec{r}_2$ (given)

$\lambda = \dfrac{2}{3}$ & $\mu = -\dfrac{1}{3}$

$\therefore$ p.v. of ' O ' are $\dfrac{2\vec{b}+\vec{d}}{3}$

Given $\left|\vec{b} \times \vec{d}\right| = 12$

Area of quad. OMCD $= \frac{1}{2}\left|\vec{d}_1 \times \vec{d}_2\right|$

$$= \frac{1}{2}\left|\left(\vec{d} - \frac{2\vec{b}+\vec{d}}{2}\right) \times \left(\frac{2\vec{b}+\vec{d}}{3} - \left(\vec{b}+\vec{d}\right)\right)\right|$$

$$= \frac{1}{12}\left|5\vec{b} \times \vec{d}\right| = 5 \text{ sq. units}$$

10. SCALAR TRIPLE PRODUCT :

Scalar triple product, **a . (b × c)**. Since the cross product **b × c** is itself a vector, we may form with it and a third vector **a** the scalar product **a .** (**b** × **c**), which is a number. Such produts of three vectors are of frequent occurrence, and we shall find it useful to examine their properties. Consider the parallelepied whose concurrent edges OA, OB, OC have the lengths and directions of the vectors **a, b, c** respectively. Then the vector **b** × **c**, which we may denote by **n**, is perpendicular to the face OBDC, and its modulus n is the measure of the area of that face. If θ is the angle between the directions of **n** and **a**, the triple product

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = an \cos\theta = \pm V, \qquad (1)$$

where V is the measure of the volume of the parallelepiped. The triple product is positive if θ is acute, that is if **a, b, c** form a right-handed system of vectors.

The same reasoning shows that each of the products **b .** (**c** × **a**) and **c .** (**a** × **b**) has the same value ± V, being positive if the system **a, b, c** is right handed, negative if left-handed. The cyclic order **a, b, c** is maintained in each of these. If, however that order is changed, the sign of the product is changed; for **b** × **c** = – **c** × **b**. Thus

$$\pm V = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = -(\mathbf{c} \times \mathbf{b}) \cdot \mathbf{a} = -\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b})$$

$$\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} = -(\mathbf{a} \times \mathbf{c}) \cdot \mathbf{b} = -\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})$$

$$\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = -(\mathbf{b} \times \mathbf{a}) \cdot \mathbf{c} = -\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a}).$$

Thus the value of the product depends on the cyclic order of the factors, but is independent of the position of the dot and cross. These may be interchanged at pleasure. It is usual to denote the above product by [**abc**] or [**a, b, c**], which indicate the three factors and their cyclic order. Then

$$[\mathbf{abc}] = -[\mathbf{acb}]. \qquad (2)$$

If the three vectors are coplanar their scalar triple product is zero. For **b** × **c** is then perpendicular to **a**, and their scalar product vanshes. Thus the vanishing of [**abc**] is the condition that the vectors should be coplanar. If two of the vectors are parallel this condition is satisfied. In particular, if two of them are equal the product is zero.

There is a very simple and covenient expression for the product [**abc**] in terms of rectangular components of the vectors.

With the usual notation we have

$$\mathbf{b} \times \mathbf{c} = (b_2c_3 - b_3c_2,\ b_3c_1 - b_1c_3 \ .\ b_1 c_2 - b_2c_1),$$

and $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \qquad (3)$$

This is the well known expression for the volume of a parallelepiped with one corner at the origin. More generally, if in terms of three non-coplanar vectors **l, m, n** we write

$$\mathbf{a} = a_1\mathbf{l} + a_2\mathbf{m} + a_3\mathbf{n},$$

and so on, it is easily shown that

$$[\mathbf{abc}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [\mathbf{lmn}].$$

The product [**ijk**], of three rectangular unit vectors, is obviously equal to unity.

Lastly, since the distributive law holds for both scalar and vector products, it holds for the scalar triple product. For instance

$$[\mathbf{a}, \mathbf{b} + \mathbf{d}, \mathbf{c} + \mathbf{e}] = [\mathbf{abc}] + [\mathbf{abe}] + [\mathbf{adc}] + [\mathbf{ade}],$$

the cyclic order of the factors being preserved in each term.

Note :

(i) The scalar triple product of three vectors $\vec{a}, \vec{b}$ & $\vec{c}$ is defined as :

$\vec{a} \times \vec{b} . \vec{c} = |\vec{a}||\vec{b}||\vec{c}| \sin\theta \cos\phi$ where θ is the angle between $\vec{a}$ & $\vec{b}$ & ϕ is the angle between $\vec{a} \times \vec{b}$ & $\vec{c}$. It is also defined as $[\vec{a}\ \vec{b}\ \vec{c}]$, spelled as box product .

(ii) Scalar triple product geometrically represents the volume of the parallelopiped whose three coterminous edges are represented by $\vec{a}, \vec{b}$ & $\vec{c}$ i.e. $V = [\vec{a}\ \vec{b}\ \vec{c}]$

(iii) In a scalar triple product the position of dot & cross can be interchanged i.e.

$\vec{a} . (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) . \vec{c}$ OR $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$

(iv) $\vec{a} . (\vec{b} \times \vec{c}) = -\vec{a} . (\vec{c} \times \vec{b})$ i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\ \vec{c}\ \vec{b}]$

(v) If $\vec{a} = a_1 i + a_2 j + a_3 k$; $\vec{b} = b_1 i + b_2 j + b_3 k$ & $\vec{c} = c_1 i + c_2 j + c_3 k$ then $[\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

In general , if $\vec{a} = a_1\vec{l} + a_2\vec{m} + a_3\vec{n}$; $\vec{b} = b_1\vec{l} + b_2\vec{m} + b_3\vec{n}$ & $\vec{c} = c_1\vec{l} + c_2\vec{m} + c_3\vec{n}$

then $\left[\vec{a}\vec{b}\vec{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \left[\vec{l}\,\vec{m}\,\vec{n}\right]$; where $\vec{\ell}, \vec{m}$ & $\vec{n}$ are non coplanar vectors .

(vi) If $\vec{a}, \vec{b}, \vec{c}$ are coplanar $\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$.

(vii) Scalar product of three vectors, two of which are equal or parallel is 0 i.e. $[\vec{a}\vec{b}\vec{c}] = 0$,

(viii) If $\vec{a}, \vec{b}, \vec{c}$ are non – coplanar then $[\vec{a}\ \vec{b}\ \vec{c}] > 0$ for right handed system & $[\vec{a}\ \vec{b}\ \vec{c}] < 0$ for left handed system .

(ix) $[K\vec{a}\ \vec{b}\ \vec{c}] = K[\vec{a}\ \vec{b}\ \vec{c}]$

(x) $[(\vec{a} + \vec{b})\ \vec{c}\ \vec{d}] = [\vec{a}\ \vec{c}\ \vec{d}] + [\vec{b}\ \vec{c}\ \vec{d}]$

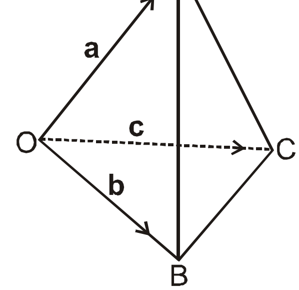
(xi) $\left[\vec{a} - \vec{b} \quad \vec{b} - \vec{c} \quad \vec{c} - \vec{a}\right] = 0$ & $\left[\vec{a} + \vec{b} \quad \vec{b} + \vec{c} \quad \vec{c} + \vec{a}\right] = 2\left[\vec{a}\ \vec{b}\ \vec{c}\right]$.

Tetrahedron. With one vertex O as origin, let the other vertices A, B, C be the points **a, b, c** respectively. Then the vector area of OBC is $\frac{1}{2}\mathbf{b} \times \mathbf{c}$, and the value of the tetrahedron in

$$V = |\frac{1}{3}\mathbf{a} . (\frac{1}{2}\mathbf{a} \times \mathbf{c})| = \frac{1}{6}|[\mathbf{abc}]|.$$

Suppose we require the length p of the common perpendicular to the two edges AB, OC. The directions of these lines are theose of the vectors **b** – **a** and **c**, while **a**, **c** are two points, one on each line. If θ is their angle of inclination,

$$p = \frac{[\mathbf{b} - \mathbf{a}, \mathbf{c}, \mathbf{a} - \mathbf{c}]}{AB.OC.\sin\theta}.$$



The numerator of this expression reduces to [**abc**]. Hence the relation.

$$V = \frac{1}{6} AB \,.\, OC \,.\, |\,p\,|.\, \sin\theta$$

The volume of a tetrahedron whose vertices are the points **a, b, c, d** is the modulus of $\frac{1}{6}$ [**a** – **d**, **b** – **d**, **c** – **d**].

The positon vector of the centroid of a tetrahedron if the pv's of its angular vertices are $\vec{a}, \vec{b}, \vec{c}$ & $\vec{d}$ are given by $\frac{1}{4}[\vec{a} + \vec{b} + \vec{c} + \vec{d}]$.

Note that this is also the point of concurrency of the lines joining the vertices to the centroids of the opposite faces and is also called the centre of the tetrahedron. In case the tetrahedron is regular it is equidistant from the vertices and the four faces of the tetrahedron .

Ex.25 If $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, b_1\hat{i} + b_2\hat{j} + b_3\hat{k}, c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ are three mutually perpendicular unit vectors, prove that $a_1\hat{i} + b_1\hat{j} + c_1\hat{k}, a_2\hat{i} + b_2\hat{j} + c_2\hat{k}, a_3\hat{i} + b_3\hat{j} + c_3\hat{k}$ are also mutually perpendicular unit vectors.

Sol. Let the three given unit vectors be $\hat{a}, \hat{b}$ and $\hat{c}$. Since they are mutually perpendicular $\hat{a}\,.\,(\hat{b} \times \hat{c}) = 1$.

$$\Rightarrow \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 1 \Rightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 1$$

$\Rightarrow \quad a_1\hat{i} + b_1\hat{j} + c_1\hat{k}, a_2\hat{i} + b_2\hat{j} + c_2\hat{k}, a_3\hat{i} + b_3\hat{j} + c_3\hat{k}$ are mutually perpendicular.

Ex.26 If four forces are acting at a point are in equilibrium then prove that their vector sum is zero.

Sol. Let **a, b, c, d** be unit vectors in the directions of the forces, and F_1,...., F_4 the miasures of the forces. Then

$$F_1\mathbf{a} + F_2\mathbf{b} + F_3\mathbf{c} + F_4\mathbf{d} = \mathbf{0}. \qquad (1)$$

If we multiply throughout scalarly by **c** × **d**, two of the terms disappear containing triple products with a repeated factor, and disapear containing triple products with a repeated factor, and we find

$$F_1[\mathbf{acd}] + F_2[\mathbf{bcd}] = 0.$$

Similarly, on forming the scalar product of the first member of (1) with **a** × **c**, we have

$$F_2[\mathbf{bac}] + F_4[\mathbf{acd}] = 0.$$

From these and a third equation which may be derived in the same way, we find

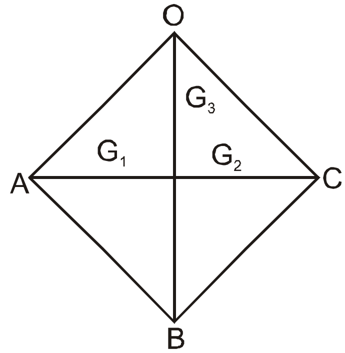
$$\frac{F_1}{[\mathbf{bcd}]} = \frac{F_2}{[\mathbf{cad}]} = \frac{F_3}{[\mathbf{bbd}]} = \frac{-F_4}{[\mathbf{abc}]}.$$

Thus each force is proportional to the scalar triple product of unit vectors in the directions of the other three, and therefore to the volume of the parallelepiped determined by those vectors. This theorem is usually attributed to Rankine.

Ex.27 If V be the volume of a tetrahedron & V′ be the volume of the tetrahedron formed by the centroids then find the ratio of V & V′

Sol. $\vec{V} = \frac{1}{6}\left[\vec{a}\ \vec{b}\ \vec{c}\right]$;

$\vec{G} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}$; $\vec{G}_1 = \frac{\vec{a}+\vec{b}}{3}$; $\vec{G}_2 = \frac{\vec{b}+\vec{c}}{3}$; $\vec{G}_3 = \frac{\vec{a}+\vec{c}}{3}$



$\overrightarrow{G_1G} = \frac{\vec{c}}{3}$; $\overrightarrow{G_2G} = \frac{\vec{a}}{3}$; $\overrightarrow{G_3G} = \frac{\vec{b}}{3}$

Hence V' = volume of tetrahedron $G G_1 G_2 G_3$

$$= \frac{1}{6}\left[\overrightarrow{GG_1}\ \overrightarrow{GG_2}\ \overrightarrow{GG_3}\right] = \frac{1}{6}\cdot\frac{1}{27}\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \frac{V}{27}$$

11. VECTOR TRIPLE PRODUCT :

Consider next the cross product of **a** and **b** × **c**, viz.

$$\mathbf{P} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}).$$

This is a vector perpendicular to both **a** and **b** × **c**. But **b** × **c** is normal to the plane of **b** and **c**, so that **P** must lie in this plane. It is therefore expressible in termsof **b** and **c** in the form

$$\mathbf{P} = l\mathbf{b} + m\mathbf{c}.$$

To find the actual expression for **P** consider unit vectors **j** and **k**, the first parallel to **b** and the second perpendicular to it in the plane, **b**, **c**. Then we may put

$$\mathbf{b} = b\mathbf{j},$$
$$\mathbf{c} = c_2\mathbf{j} + c_3\mathbf{k}.$$

In terms of **j, k** and the other unti vector **i** of the right-handed system, the remaining vector **a** may be written

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

Then $\mathbf{b} \times \mathbf{c} = bc_3\mathbf{i}$, and the triple product

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = a_3bc_3\mathbf{j} - a_2bc_3\mathbf{k}.$$
$$= (a_2c_2 + a_3c_3)\, b\mathbf{j} - a_2b(c_2\mathbf{j} + c_3\mathbf{k})$$
$$= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}. \qquad (1)$$

This is the required expression for **P** in terms of **b** and **c**.

Similarly the triple product

$$(\mathbf{b} \times \mathbf{c}) \times \mathbf{a} = -\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$
$$= (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c} - (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b}. \qquad (2)$$

It will be noticed that the expansios(1) and (2) are both written down by the same rule. Each scalar product involves the factor outside the bracket; and the first is the scalar product of the extremes.

In a vector triple product the position of the brackets cannot be changed without altering the value of the product. For (**a** × **b**) × **c** is a vector expressible in terms of **a** and **b**; **a** × (**b** × **c**) is one expressible in terms of **b** and **c**. The products in general therefore represent different vectors.

If a vector **r** is resoved into two others in the plane of **a** and **r**, one parallel to **a** and the other perpendicular to it, the former is $\frac{\mathbf{a.r}}{a^2}\,\mathbf{a}$, and therefore the latter $\frac{\mathbf{a.r}}{a^2} = \frac{(\mathbf{a.a})\mathbf{r} - (\mathbf{a.r})\mathbf{a}}{a^2} = \frac{\mathbf{a} \times (\mathbf{r} \times \mathbf{a})}{a^2}$

Geometrical Interpretation of $\vec{a} \times (\vec{b} \times \vec{c})$

Consider the expression $\vec{a} \times (\vec{b} \times \vec{c})$ which itself is a vector, since it is a cross product of two vectors $\vec{a}\ \&\ (\vec{b} \times \vec{c})$. Now $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector perpendicular to the plane containing $\vec{a}\ \&\ (\vec{b} \times \vec{c})$ but $\vec{b} \times \vec{c}$ is a vector perpendicular to the plane $\vec{b}\ \&\ \vec{c}$, therefore $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector lies in the plane of $\vec{b}\ \&\ \vec{c}$ and perpendicular to $\vec{a}$. Hence we can express $\vec{a} \times (\vec{b} \times \vec{c})$ in terms of $\vec{b}\ \&\ \vec{c}$ i.e. $\vec{a} \times (\vec{b} \times \vec{c}) = x\vec{b} + y\vec{c}$ where x & y are scalars .

Note :

(i) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

(ii) $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$

(iii) $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$

Ex.28 Find a vector $\vec{v}$ which is coplanar with the vectors $\hat{i}+\hat{j}-2\hat{k}$ & $\hat{i} - 2\hat{j} + \hat{k}$ and is orthogonal to the vector $-2\hat{i} + \hat{j} + \hat{k}$. It is given that the projection of $\vec{v}$ along the vector $\hat{i} - \hat{j} + \hat{k}$ is equal to $6\sqrt{3}$.

Sol. A vector coplanar with $\vec{a}$ & $\vec{b}$ and orthogonal to $\vec{c}$ is parallel to the triple product , $\left(\vec{a} \times \vec{b}\right) \times \vec{c}$

$= \left(\vec{a} \cdot \vec{c}\right) \vec{b} - \left(\vec{b} \cdot \vec{c}\right) \vec{a}$

Hence $\vec{v} = \alpha \left[-3\left(\hat{i}-2\hat{j}+\hat{k}\right)+3\left(\hat{i}+\hat{j}-2\hat{k}\right)\right] = 9\alpha \left(\hat{j} - \hat{k}\right)$

Projection of $\vec{v}$ along $\hat{i} - \hat{j} + \hat{k} = \dfrac{\vec{v} \cdot \left(\hat{i} - \hat{j} + \hat{k}\right)}{\left|\vec{i} - \vec{j} + \vec{k}\right|} = 6\sqrt{3}$

$9\alpha \left(\hat{j} - \hat{k}\right) \cdot (\hat{i} - \hat{j} + \hat{k}) = 18 \quad 9\alpha(-1-1) = 18 \quad \Rightarrow \alpha = -1$ Ans. : $9\left(-\hat{j} + \hat{k}\right)$

Ex.29 ABCD is a tetrahedron with A (– 5, 22, 5); B (1, 2, 3); C (4, 3, 2); D (– 1, 2, – 3). Find $\vec{AB} \times \left(\vec{BC} \times \vec{BD}\right)$. What can you say about the values of $\left(\vec{AB} \times \vec{BC}\right) \times \vec{BD}$ and $\left(\vec{AB} \times \vec{BD}\right) \times \vec{BC}$. Calculate the volume of the tetrahedron ABCD and the vector area of the triangle AEF where the quadrilateral ABDE and quadrilateral ABCF are parallelograms.

Sol. $\vec{AB} \times \left(\vec{BC} \times \vec{BD}\right) = 0$; $\left(\vec{AB} \times \vec{BC}\right) \times \vec{BD} = 0$; $\left(\vec{AB} \times \vec{BD}\right) \times \vec{BC} = 0$;

Note that $\vec{AB}$; $\vec{BC}$; $\vec{BD}$ are mutually perpendicular .

Volume $= \dfrac{1}{6} [\vec{AB}, \vec{BC}, \vec{BD}] = \dfrac{220}{3}$ cu. units

Vector area of triangle AEF $= \dfrac{1}{2} \vec{AF} \times \vec{AE} = \dfrac{1}{2} \vec{BC} \times \vec{BD} = -3\hat{i} + 10\hat{j} + \hat{k}$

12. PRODUCTS OF FOUR VECTORS

Scalar product of four vectors. The products already considered are usually sufficient for parctical applications. But we occasionally meet with products of four vectors of the following types.

Consider the scalar product of **a** × **b** and **c** × **d**. This is a number easily expressible in terms of the scalar products of the individual vectors. For, in virtue of the fact that in a scalar triple product the dot and cross may be interchanged, we may write

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \mathbf{a} \cdot \mathbf{b} \times (\mathbf{c} \times \mathbf{d})$$
$$= \mathbf{a} \cdot ((\mathbf{b} \cdot \mathbf{d})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{d})$$
$$= (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}).$$

Writing this result in the form of a determinant, we have

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{a.c} & \mathbf{a.d} \\ \mathbf{b.c} & \mathbf{b.d} \end{vmatrix}.$$

Vector product of four vectors. Consider next the vector product of **a** × **b** and **c** × **d**. This is a vector at right angles to **a** × **b**, and therefore coplanar with **a** and **b**. Similarly it is coplanar with **c** and **d**. It must therefore be parallel to the line of intersection of a plane parallel to **a** and **b** with another parallel to **c** and **d**.

To express the product in (**a** × **b**) × (**c** × **d**) in terms of **a** and **b**, regard it as the vector triple product of **a, b** and **m**, where **m** = **c** × **d**. Then

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\times\mathbf{b})\times\mathbf{m} = (\mathbf{a}\,.\,\mathbf{m})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{m})\mathbf{a}$$
$$= [\mathbf{acd}]\mathbf{b} - [\mathbf{bcd}]\mathbf{a}. \qquad (1)$$

Similarly, regarding it as the vector product of **n, c** and **d**, where **n** = **a** × **b**, we may write it

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = \mathbf{n}\times(\mathbf{c}\times\mathbf{d}) = (\mathbf{n}\,.\,\mathbf{d})\mathbf{c} - (\mathbf{n}\,.\,\mathbf{c})\mathbf{d}$$
$$= [\mathbf{abd}]\mathbf{c} - [\mathbf{abc}]\mathbf{d}. \qquad (2)$$

Equating these two expressions we have a relation between the four vectors **a, b, c, d** viz.

$$[\mathbf{bcd}]\mathbf{a} - [\mathbf{acd}]\mathbf{b} + [\mathbf{abd}]\mathbf{c} - [\mathbf{abc}]\mathbf{d} = \mathbf{0}. \qquad (3)$$

Writing **r** instead of **d**, we may express any vector **r** in terms of three other vectors **a, b, c** in the form

$$\mathbf{r} = \frac{[\mathbf{rbc}]\mathbf{a} + [\mathbf{rca}]\mathbf{b} + [\mathbf{rab}]\mathbf{c}}{[\mathbf{abc}]}, \qquad (4)$$

which is valid except when the denominator [**abc**] vanishes, that is except when **a, b, c** are coplanar.

Ex.30 Show that , $(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a}) = [\vec{a}\ \vec{b}\ \vec{c}]\,\vec{c}$ and deduce that ,

$$[\vec{a}\times\vec{b}\ ,\ \vec{b}\times\vec{c}\ ,\ \vec{c}\times\vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]^2.$$

Sol. L.H.S. : $(\vec{b}\times\vec{c})\times\vec{u} = (\vec{b}\,.\,\vec{u})\,\vec{c} - (\vec{c}\,.\,\vec{u})\,\vec{b}$ $\qquad (\vec{u} = \vec{c}\times\vec{a})$

$$= [\vec{b}\ \vec{c}\ \vec{a}]\,\vec{c} - 0$$

Hence $(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a}) = [\vec{a}\ \vec{b}\ \vec{c}]\ \vec{c}$

taking dot with $\vec{a}\times\vec{b}$, $\quad [\vec{a}\times\vec{b}\ ,\ \vec{b}\times\vec{c}\ ,\ \vec{c}\times\vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]^2.$

Ex.31 Show that $\vec{a}\times((\vec{q}\times\vec{c})\times(\vec{p}\times\vec{b})) = \vec{b}\times((\vec{p}\times\vec{c})\times(\vec{q}\times\vec{a})) + \vec{c}\times((\vec{p}\times\vec{a})\times(\vec{q}\times\vec{b}))$

Sol. consider $\vec{a}\times[\underbrace{(\vec{q}\times\vec{c})}_{\vec{u}}\times(\vec{p}\times\vec{b})] = \vec{a}\times[(\vec{u}\,.\,\vec{b})\vec{p} - (\vec{u}\,.\,\vec{p})\vec{b}]$

$$= (\vec{a}\times\vec{p})\,.\,[\vec{q}\ \vec{c}\ \vec{b}] - (\vec{a}\times\vec{b})\,.\,[\vec{q}\ \vec{c}\ \vec{b}] \qquad \text{— (1)}$$

similarly $\vec{b}\times[(\vec{p}\times\vec{c})\times(\vec{q}\times\vec{a})] = (\vec{b}\times\vec{q})\,.\,[\vec{p}\ \vec{c}\ \vec{a}] - (\vec{b}\times\vec{a})\,.\,[\vec{p}\ \vec{c}\ \vec{q}]$ — (2)

and $\vec{c}\times[\underbrace{(\vec{p}\times\vec{a})}_{\vec{u}}\times\underbrace{(\vec{q}\times\vec{b})}_{\vec{v}}] = \vec{c}\times[\vec{u}\times\vec{v}]$

$$= (\vec{c}\,.\,\vec{v})\,\vec{u} - (\vec{c}\,.\,\vec{u})\,\vec{v} = [\vec{c}\ \vec{q}\ \vec{b}](\vec{p}\times\vec{a}) - [\vec{c}\ \vec{p}\ \vec{a}]\times(\vec{q}\times\vec{b}) \qquad \text{— (3)}$$

Now (1) – (2) – (3) = 0 $\Rightarrow$ result

13. RECIPROCAL SYSTEM OF VECTORS :

If $\vec{a}, \vec{b}, \vec{c}$ & $\vec{a}', \vec{b}', \vec{c}'$ are two sets of non coplanar vectors such that $\vec{a}.\vec{a}' = \vec{b}.\vec{b}' = \vec{c}.\vec{c}' = 1$ then the two systems are called Reciprocal System of vectors.

In terms of the vectors, **a**′, **b**′, **c**′ are defined by the equations

$$\mathbf{a}' = \frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]}; \qquad \mathbf{b}' = \frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{abc}]}; \qquad \mathbf{c}' = \frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{abc}]}$$

They also possess the property that the scalar product of any other pair of vectors, one from each system, is zero. for instance,

$$\mathbf{a} \,.\, \mathbf{b}' = [\mathbf{aca}] / [\mathbf{abc}] = 0,$$

since, in the numerator, two factors of the scalar triple product are equal. Similarly $\mathbf{c} \,.\, \mathbf{b}' = 0$.

The symmetry of the above relations suggests that if $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ is the reciprocal system to **a, b, c** then **a, b, c** is the reciprocal system to $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$; and this may be easily proved. For

$$\mathbf{b}' \times \mathbf{c}' = (\mathbf{c} \times \mathbf{a}) \times (\mathbf{a} \times \mathbf{b}) / [\mathbf{abc}]^2 = [\mathbf{cab}]\mathbf{a} / [\mathbf{abc}]^2,$$

$$\mathbf{a}'.\, \mathbf{b}' \times \mathbf{c}' = \mathbf{a} \,.\, \mathbf{b} \times \mathbf{c} / [\mathbf{abc}]^2 = 1/[\mathbf{abc}],$$

and therefore $\mathbf{b}' \times \mathbf{c}' / [\mathbf{a}'\mathbf{b}'\mathbf{c}'] = \mathbf{a}$. Similar expressions may be found for **b** and **c**, showing that **a, b, c** is reciprocal system to $\mathbf{a}', \mathbf{b}', \mathbf{c}'$. Incidentally we have shown that $[\mathbf{a}'\mathbf{b}'\mathbf{c}']$ is the reciprocal of $[\mathbf{abc}]$, so that these have the same sign, and the two systems of vectors are either both right-handed or both left-handed. We also have the formula $\mathbf{r} = (\mathbf{r} \,.\, \mathbf{a})\mathbf{a}' + (\mathbf{r} \,.\, \mathbf{b})\mathbf{b}' + (\mathbf{r} \,.\, \mathbf{c})\mathbf{c}'$.

The rectangular system of unit vectors **i, j, k** is its own reciprocal.

Ex.32 If $\vec{a}', \vec{b}', \vec{c}'$ is a reciprocal system of $\vec{a}, \vec{b}, \vec{c}$ show that

(i) $\vec{a}\times\vec{a}' + \vec{b}\times\vec{b}' + \vec{c}\times\vec{c}' = 0$

(ii) $[\vec{a}\,\vec{b}\,\vec{c}]\,[\vec{a}'\,\vec{b}'\,\vec{c}'] = 1$

Sol. We know that

$$\vec{a}' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}, \vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]} \text{ and } \vec{c}' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

$\Rightarrow$ $\vec{a}\times\vec{a}' + \vec{b}\times\vec{b}' + \vec{c}\times\vec{c}'$

$$= \frac{\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b})}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Now $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\,.\,\vec{c})\vec{b} - (\vec{a}\,.\,\vec{b})\vec{c}\,.\;\vec{b}\times(\vec{c}\times\vec{a})$

$= (\vec{b}\,.\,\vec{a})\vec{c} - (\vec{b}\,.\,\vec{c})\vec{a}$

and $\vec{c}\times(\vec{a}\times\vec{b}) = (\vec{b}\,.\,\vec{c})\vec{a} - (\vec{a}\,.\,\vec{c})\vec{b}$

$\therefore$ $\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = 0$

So $\vec{a}\times\vec{a}' + \vec{b}\times\vec{b}' + \vec{c}\times\vec{c}' = 0$

Next $[\vec{a}'\,\vec{b}'\,\vec{c}'] = (\vec{a}'\times\vec{b}').\vec{c}'$

$$= \frac{1}{[\vec{a}\,\vec{b}\,\vec{c}]^3}[(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})].(\vec{a}\times\vec{b})$$

$$= \frac{1}{[\vec{a}\,\vec{b}\,\vec{c}]^3}[(\vec{e}\,.\,\vec{a})\vec{c} - (\vec{e}\,.\,\vec{c})\vec{a}]\,.\,(\vec{a}\times\vec{b})\ [\vec{e} = \vec{b}\times\vec{c}]$$

$$= \frac{1}{[\vec{a}\,\vec{b}\,\vec{c}]^3}[\{(\vec{b}\times\vec{c})\,.\,\vec{a}\}\,\vec{c}]\,.\,(\vec{a}\times\vec{b})$$

$$= \frac{1}{[\vec{a}\,\vec{b}\,\vec{c}]^3}[(\vec{b}\times\vec{c})\,.\,\vec{a}]\,[\vec{c}\,.\,(\vec{a}\times\vec{b})]$$

$$= \frac{1}{[\vec{a}\,\vec{b}\,\vec{c}]^3}[\vec{a}.(\vec{b}\times\vec{c})]\,[(\vec{a}\times\vec{b}).\vec{c}] = \frac{1}{[\vec{a}\,\vec{b}\,\vec{c}]}.$$

14. VECTOR EQUATIONS

Ex.33 Sove the equation $\mathbf{x} \times \mathbf{a} = \mathbf{b}$, $(\mathbf{a} \,.\, \mathbf{b} = 0)$.

Sol. Form the vector product of each member with **a**, and obntain

$$\mathbf{a}^2\mathbf{x} - (\mathbf{a} \cdot \mathbf{x})\, \mathbf{a} = \mathbf{a} \times \mathbf{b},$$

The general solution, with λ as parameter, is

$$\mathbf{x} = \lambda\mathbf{a} + \mathbf{a} \times \mathbf{b}/\mathbf{a}^2.$$

Ex.34 Solve the simultaneous equations

$$p\mathbf{x} + q\mathbf{y} = \mathbf{a},\ \mathbf{x} \times \mathbf{y} = \mathbf{b}, \qquad (\mathbf{a} \cdot \mathbf{b} = 0).$$

Sol. Multiply the first vectorially by $\mathbf{x}$, and substitute for $\mathbf{x} \times \mathbf{y}$ from the second. Then $q\mathbf{b} = \mathbf{x} \times \mathbf{a}$, which is of the same form as the equation in the preceding example. Thus

$$\mathbf{x} = \lambda\mathbf{a} + q\mathbf{a} \times \mathbf{b}/\mathbf{a}^2.$$

Substitution of this value in the first equation gives y.

Ex.35 Find $\mathbf{x}$ so as to satisfy both the equations

$$\mathbf{x} \times \mathbf{a} = \mathbf{b}, \qquad \mathbf{x} \cdot \mathbf{c} = p \qquad (\mathbf{a} \cdot \mathbf{b} = 0) \qquad \text{(i)}$$

Sol. Mulitply the first vectorially by $\mathbf{c}$, expand, and use the second. Then

$$(\mathbf{c} \cdot \mathbf{a})\mathbf{x} - p\mathbf{a} = \mathbf{c} \times \mathbf{b}.$$

Thus $\quad \mathbf{x} = (p\mathbf{a} + \mathbf{c} \times \mathbf{b})/\mathbf{a} \cdot \mathbf{c} \qquad$ (ii)

provided $\mathbf{a} \cdot \mathbf{c} \neq 0$. If, however, $\mathbf{a} \cdot \mathbf{c} = 0$, use the general solution of the first equation (i).

$$\mathbf{x} = \lambda\mathbf{a} + \mathbf{a} \times \mathbf{b}/\mathbf{a}^2 \qquad \text{(iii)}$$

This will satisfy the second equatin (i) for any value of λ provided $p\mathbf{a}^2 = \mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$, which is a necessary condition when $\mathbf{a} \cdot \mathbf{c} = 0$.

Ex.36 Solve $\quad \mathbf{x} \times \mathbf{a} + (\mathbf{x} \cdot \mathbf{b})\mathbf{c} = \mathbf{d} \qquad$ (i)

Sol. Multiply scalarly by $\mathbf{a}$. Then $(\mathbf{x} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{c}) = \mathbf{a} \cdot \mathbf{d}$. Subsitute for $\mathbf{x} \cdot \mathbf{b}$ in (i) and obtain

$$\mathbf{x} \times \mathbf{a} = \mathbf{d} - (\mathbf{a} \cdot \mathbf{d})\mathbf{c}/\mathbf{a} \cdot \mathbf{c} = \mathbf{a} \times (\mathbf{d} \times \mathbf{c})/\mathbf{a} \cdot \mathbf{c}.$$

$$\mathbf{x} = \lambda\mathbf{a} + \mathbf{a} \times (\mathbf{a} \times (\mathbf{d} \times \mathbf{c}))\ /\ (\mathbf{a} \cdot \mathbf{c})\mathbf{a}^2.$$

Ex.37 Solve $p\mathbf{x} + \mathbf{x} \times \mathbf{a} = \mathbf{b}$, $(p \neq 0) \qquad$ (i)

Sol. Multiply scalarly by $\mathbf{a}$. Then

$$p\mathbf{x} \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{a} \qquad \text{(ii)}$$

Multiply (i) vectorially by $\mathbf{a}$, expand the triple product, and substitute for $\mathbf{x} \cdot \mathbf{a}$ from (ii). Then

$$p^2\mathbf{x} \times \mathbf{a} + (\mathbf{b} \cdot \mathbf{a})\mathbf{a} - p\mathbf{a}^2\mathbf{x} = p\mathbf{b} \times \mathbf{a}.$$

Eliminate $\mathbf{x} \times \mathbf{a}$ between this equation and (i), and find

$$\mathbf{x} = (p^2\mathbf{b} + (\mathbf{b} \cdot \mathbf{a})\mathbf{a} - p\mathbf{a}^2\,\mathbf{x} = p\mathbf{b} \times \mathbf{a}.$$

Ex.38 If $\vec{A} + \vec{B} = \vec{a}$, $\vec{A} \cdot \vec{a} = 1$ and $\vec{A} \times \vec{B} = \vec{b}$ then prove that

$$\vec{A} = \frac{\vec{a} \times \vec{b} + \vec{a}}{|\vec{a}|^2} \quad \text{and} \quad \vec{B} = \frac{\vec{b} \times \vec{a} + \vec{a}\left(|\vec{a}|^2 - 1\right)}{|\vec{a}|^2}$$

Sol. $\vec{A} + \vec{B} = \vec{a} \qquad$ taking dot with $\vec{a}$

$$\vec{a} \cdot \vec{B} = |\vec{a}|^2 - 1 \qquad \text{— (1)}$$

$\vec{A} \times \vec{B} = \vec{b} \qquad$ taking cross with $\vec{a}$

$$\left(\vec{a} \cdot \vec{B}\right)\vec{A} - \left(\vec{a} \cdot \vec{A}\right)\vec{B} = \vec{a} \times \vec{b}$$

$$\left(|\vec{a}|^2 - 1\right)\vec{A} - \vec{B} = \vec{a} \times \vec{b} \qquad \text{— (2)}$$

Solving (2) and $\vec{A} + \vec{B} = \vec{a}$ simultaneously we get the desired result.

Ex.39 Solve the vector equation in $\vec{x}$: $\quad \vec{x} + \vec{x} \times \vec{a} = \vec{b}$.

Sol. Taking dot with $\vec{a} \qquad \vec{x} \cdot \vec{a} = \vec{b} \cdot \vec{a} \qquad$ — (1)

Taking cross with $\vec{a}$ $\quad \vec{x} \times \vec{a} + (\vec{x} \times \vec{a}) \times \vec{a} = \vec{b} \times \vec{a}$ — (2)

$\vec{b} - \vec{x} + (\vec{x} \cdot \vec{a})\vec{a} - (\vec{a} \cdot \vec{a})\vec{z} = \vec{b} \times \vec{a}$

$$\vec{b} + \vec{a}(\vec{b} \cdot \vec{a}) - \vec{b} \times \vec{a} = \vec{x}(1 + \vec{a} \cdot \vec{a})$$

Ans. : $\vec{x} = \dfrac{1}{1 + \vec{a} \cdot \vec{a}} \left\{\vec{b} + (\vec{a} \cdot \vec{b})\vec{a} + \vec{a} \times \vec{b}\right\}$

Ex.40 Express a vector $\vec{R}$ as a linear combination of a vector $\vec{A}$ and another perpendicular to $\vec{A}$ and coplanar with $\vec{R}$ and $\vec{A}$.

Sol. $\vec{A} \times (\vec{A} \times \vec{R})$ is a vector perpendicular to $\vec{A}$ and coplanar with $\vec{A}$ and $\vec{R}$.

Hence let , $\vec{R} = \lambda \vec{A} + \mu \vec{A} \times (\vec{A} \times \vec{R})$ — (1)

taking dot with $\vec{A}$, $\quad \vec{R} \cdot \vec{A} = \lambda \vec{A} \cdot \vec{A} \quad \Rightarrow \lambda = \dfrac{\vec{R} \cdot \vec{A}}{\vec{A} \cdot \vec{A}}$

again taking cross with $\vec{A}$,

$$\vec{R} \times \vec{A} = \mu \left[\vec{A} \times (\vec{A} \times \vec{R})\right] \times \vec{A}$$

$$= \mu \left[(\vec{A} \cdot \vec{R})\vec{A} - (\vec{A} \cdot \vec{R})\vec{R}\right] \times \vec{A} \qquad = -\mu (\vec{A} \cdot \vec{A})(\vec{R} \cdot \vec{A})$$

$\therefore \mu = -\dfrac{1}{\vec{A}.\vec{A}}$ $\qquad$ Hence $\vec{R} = \left(\dfrac{\vec{R}.\vec{A}}{\vec{A}.\vec{A}}\right)\vec{A} - \dfrac{1}{\vec{A}.\vec{A}} \vec{A} \times (\vec{A} \times \vec{R})]$

15. VECTOR EQUATION OF A LINE :

Parametric vector equation of a line passing through two point $A(\vec{a})$ & $B(\vec{b})$ is given by, $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$ where t is a parameter. If the line passes through the point $A(\vec{a})$ & is parallel to the vector b then its equation is, $\vec{r} = \vec{a} + t\vec{b}$

If P is a point on this straight line the vector $\overrightarrow{AP}$ is parallel to **b**, and is therefore equal to t**b**, where t is some real number positive for points on one side of A, and negative for points on the other, varying form point to point. Thus, if **a** is the position vector of A, that of P is

$$r = \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$$
$$= \mathbf{a} + t\mathbf{b}. \qquad (1)$$

And since any point on the given straight line has a position vector given by (1) for some value of t, we may speak of (1) as a vector equation of the straight line.

To find a vector equation of the straight line passing through the points A and B, whose position vectors are **a** and **b**, we observe that $\overrightarrow{AB}$ = **b** – **a**; so that straight lines is one through the point A parallel to **b** – **a**. Its vector equation is therefore **r** = **a** + t(**b** – **a**)

or $\qquad$ **r** = (1 – t) **a** – t**b**. $\qquad$ (3)

The three points A, B, P are collinear; and if the linear equation (3) connecting their position vectors is written (1 – t)**a** + t**b** – **r** = **0**,

with all the terms on one side, the algebraic sum of the co-efficients of the vectors is zero. This is the neccessary and sufficient condition that three points should be collinear.

Ex.41 If ABCE is a parallelogram and L, M are the mid points of the sides AB and BC respectively, show that DL, AC meet in apoint of trisection and similarly DM and AC.

Sol. Let $\overrightarrow{AB} = \vec{a}, \overrightarrow{AD} = \vec{b}$ in the parallelogram ABCD

Then $\overrightarrow{AL} = \frac{\vec{a}}{2}, \overrightarrow{BM} = \frac{\vec{b}}{2}$

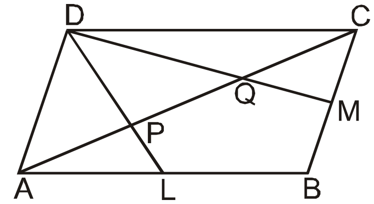
$\therefore$ let DL, DM meet AC in P and Q respectively

$\overrightarrow{AP} = \lambda\overrightarrow{AC} = \lambda(\vec{a} + \vec{b})$ ...(1)

Also $\overrightarrow{AP} = \overrightarrow{AD} + \overrightarrow{DP} = \vec{b} + \mu\overrightarrow{DL} = \vec{b} + \mu(\overrightarrow{DL} + \overrightarrow{AL})$

i.e. $\overrightarrow{AP} = \vec{b} + \mu\left(-\vec{b} + \frac{\vec{a}}{2}\right)$ ...(2)

Equating the coefficients of $\vec{a}, \vec{b}$ in (1) and (2), we get



$\lambda = \frac{\mu}{2}$, $\lambda = 1 - \mu$ $\Rightarrow$ $\lambda = \frac{1}{3}$

$\therefore$ $\overrightarrow{AP} = \frac{1}{3}\overrightarrow{AC}$ i.e. P is a point of trisection of AC.

Similarly, we can prove $\overrightarrow{AQ} = \frac{2}{3}\overrightarrow{AC}$

Bisector of the angle between two straight lines. To find the equation of the bisector of the angle between the straight lines OA and OB, parallel to the unit vectors **a** and **b** respectively, take the point O as origin and let P be any point on the bisector.

Then, if PN is drawn parallel to to AO cutting OB in N, the angles OPN and NOP are equal, and ON = NP. But these are parallel to **b** and **a** respectively; so that $\overrightarrow{ON}$ = t**b** and $\overrightarrow{NP}$ = t**a**, where t is some real number. The position vector of P is therefore **r** = t (**a** + **b**).

This is the required equation of the bisector, the value of t varying as P moves along the line.

The bisector OP′ of the supplementary angle B′OA is the bisector of the angle between straight lines whose directions are those of **a** and –**b**; and its equation is therefore **r** = t(**a** – **b**).

If **a** and **b** are not unit vectors, the equations of the above bisectors are **r** = r $\left(\frac{\mathbf{a}}{a} \pm \frac{\mathbf{b}}{b}\right)$.

Ex.42 The internal bisector of an angle A of a triangle ABC divides the side BC in the ratio AB : AC.

Sol. Let $\overrightarrow{AB}$ = **c** and $\overrightarrow{AC}$ = **b.** Then, with A as origin, the internal bisector of the angle A is the line r $= t\left(\frac{\mathbf{b}}{b} + \frac{\mathbf{c}}{c}\right) = t\left(\frac{c\mathbf{b} + b\mathbf{c}}{bc}\right)$,

where, according to our usual notation, b is the modulus of **b** and **c** is that of **c**. Giving t the value bc/(b + c) we see that the bisector passes through the point (c**b** + b**c**) /(b + c),

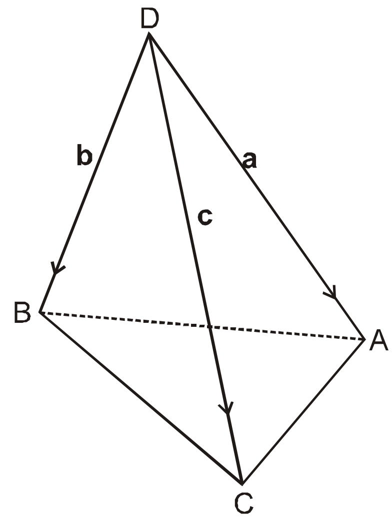
which is the centroid of the points B and C with associated numbers b and c respectively, and therefore lies in BC, dividing it in the ratio c : b or AB : AC. Hence the theorem.

Similarly the external bisector of the angle A is the straight line r = $t\left(\frac{\mathbf{c}}{b} - \frac{\mathbf{b}}{c}\right) = t\left(\frac{b\mathbf{c} - c\mathbf{b}}{bc}\right)$,

which passes through the point (b**c** – c**b**)/(b – c). This point which is the centroid of B and C with associated number b and –c, divides BC externaly in the ratio AB : AC.

Ex.43 The lines joining the vertices of tetrahedron to the centroids of the opposite faces are concurrent.

Sol. Let ABCD be the tetrahedron. Take D as origin of position vectors. Then the line joining D to the centroid of the face ABC is $\mathbf{r} = s(\mathbf{a} + \mathbf{b} + \mathbf{c})$.



Also the centroid of the face DAC is the poitn $\frac{1}{3}(\mathbf{a} + \mathbf{c})$; and the line joining this to B is $\mathbf{r} = t\mathbf{b} + (1 - t)\frac{\mathbf{a}+\mathbf{c}}{3}$.

These two lines intersect at the point for which $s = t = \frac{1}{4}$, that is the point $\frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c})$. Form the symmetry of this result the theorem follows.

Non-parametric equation of straight line. Consider the straight line through A in the direction of the unit vector e. For any point **r** on this line, the vector **r** – **a** is parallel to **e**, so that

$$(\mathbf{r} - \mathbf{a}) \times \mathbf{e} = \mathbf{0} \qquad \text{(i)}$$

is one form of the equation of the line. The perpendicular distance from a point P(**p**) to the line has magnitude AP sin θ, which is $|(\mathbf{p} - \mathbf{a}) \times \mathbf{e}|$. It is thus the magnitude of the vector obtained by substituting **p** for **r** in the first member of equation (i). The position vector of N, the foot of the perpendicular, is $\mathbf{a} + \mathbf{e}\,.\,(\mathbf{p} - \mathbf{a})\mathbf{e}$. The vector

$$\overrightarrow{PN} = \overrightarrow{PA} + \overrightarrow{AN} = \mathbf{a} - \mathbf{p} + \mathbf{e}\,.\,(\mathbf{p} - \mathbf{a})\,\mathbf{e}.$$

Ex.44 Line L_1 is parallel to vector $\vec{\alpha} = -3\hat{i} + 2\hat{j} + 4\hat{k}$ and passes through a point A(7, 6, 2) and line L_2 is parallel to a vector $\vec{\beta} = 2\hat{i} + \hat{j} + 3\hat{k}$ and passes through a point B(5, 3, 4). Now a line L_3 parallel to a vector $\vec{r} = 2\hat{i} - 2\hat{j} - \hat{k}$ intersects the lines L_1 and L_2 at points C and D respectively. Find $\left|\overrightarrow{CD}\right|$.

Sol. $\vec{r}_1 = 7\hat{i} + 6\hat{j} + 2\hat{k} + \lambda\left(-3\hat{i} + 2\hat{j} + 4\hat{k}\right)$ $\vec{r}_2 = 5\hat{i} + 3\hat{j} + 4\hat{k} + \mu\left(2\hat{i} + \hat{j} + 3\hat{k}\right)$

$\Rightarrow \overrightarrow{CD} = (3\lambda + 2\mu - 2)\hat{i} + (-2\lambda + \mu - 3)\hat{j} + (-4\lambda + 3\mu + 2)\hat{k} = r\left(2\hat{i} - 2\hat{j} - \hat{k}\right)$

$\Rightarrow \frac{3\lambda + 2\mu - 2}{2} = \frac{-2\lambda + \mu - 3}{-2} = \frac{4\lambda - 3\mu - 2}{1} \Rightarrow \lambda = 2$ and $\mu = 1$

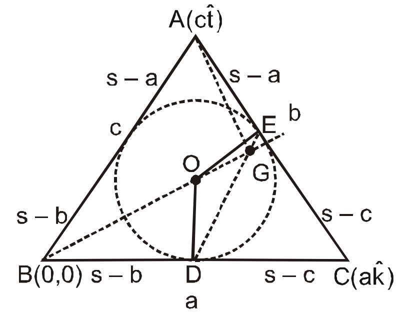
$\Rightarrow \overrightarrow{CD} = 6\hat{i} - 6\hat{j} - 3\hat{k} \Rightarrow \left|\overrightarrow{CD}\right| = 9$

Ex.45 The in-circle of the triangle ABC touches its sides at D, E, F. If O is the centre of the incircle and BO meets DE at G, use vector method to prove that AG is perpendicular to BG.

Sol. Let B be taken as the initial point. Let the position vector of C and A be $a\hat{k}$ and $c\hat{t}$ respectively where $\hat{k}$ and $\hat{t}$ are unit vectors. With normal notations of ΔABC, the positive vector of D is $(s - b)\hat{k}$ and that of E is $\frac{(s-c)c\hat{t} + (s-a)a\hat{k}}{b}$.

The equation of BO is $r = \lambda_1(\hat{t} + \hat{k})$ and that of DE is

$$r = (s-b)\hat{k} + \lambda_2\left[\frac{(s-c)c\hat{t} + (s-a)a\hat{k}}{b} - (s-b)\hat{k}\right]$$



These lines intersect at G, where

$$\lambda_1(\hat{t}+\hat{k}) = (s-b)\hat{k} + \lambda_2\left[\frac{(s-c)c\hat{t} + (s-a)a\hat{k}}{b} - (s-b)\hat{k}\right]$$

Equating the coefficients of $\hat{t}$ and $\hat{k}$, we get

$$\Rightarrow \quad \lambda_1 = \frac{(s-c)\lambda_2}{b} \text{ and } \lambda_2 = s-b+\frac{(s-a)a-(s-b)b}{b}\lambda_2.$$

$$\text{or} \quad \lambda_1 = s-b+\frac{s(a-b)-(a^2-b^2)}{(s-c)c}\lambda_1 = s-b+\frac{(a-b)-(s-a-b)}{(s-c)c}\lambda_1$$

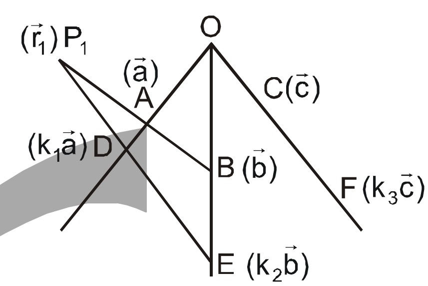
$$\Rightarrow \quad \lambda_1\left[1+\frac{a-b}{c}\right] = s-b \Rightarrow \lambda_1 = \frac{c}{2}$$

Hence position vector of G is $\frac{c}{2}(\hat{t}+\hat{k})$

$\overline{AG} = \frac{c}{2}(\hat{t}+\hat{k}) - c\hat{t} = \frac{c}{2}(\hat{k}-\hat{t})$ which is perpendicular to BG i.e. $\frac{c}{2}(\hat{t}+\hat{k})$.

Ex.46 Three concurrent straight lines OA, OB & OC are produced D, E & F respectively. Use vectors to prove that the points of intersection of AB & DE; BC & EF; CA & FD are collinear.

Sol. AB : $\vec{r} = \vec{a} + \lambda(\vec{b}-\vec{a})$



DE : $\vec{r} = k_1\vec{a} + \mu(k_2\vec{b} - k_1\vec{a})$

For the point P_1,

$(1-\lambda)\vec{a} + \lambda\vec{b} = (k_1 - k_1\mu)\vec{a} + \mu k_2\vec{b}$

$$\begin{rcases} \therefore \quad \lambda = \mu k_2 \\ \text{and} \quad 1-\lambda = k_1(1-\mu) \end{rcases} \begin{matrix} -(1) \\ -(2) \end{matrix}$$

(1) & (2) gives $\mu = \frac{1-k_1}{k_2-k_1}$; $\lambda = \frac{(1-k_1)k_2}{k_2-k_1}$

$$\therefore \vec{r}_1 = \vec{a}\left[1-\frac{(1-k_1)k_2}{k_2-k_1}\right] + \frac{(1-k_1)k_2}{k_2-k_1}\vec{b}$$

$$\text{or } \vec{r}_1 = \frac{k_1(k_2-1)}{k_2-k_1}\vec{a} - \frac{k_2(k_1-1)}{k_2-k_1}\vec{b}$$

$$\text{or } \frac{(k_2-k_1)}{(k_1-1)(k_2-1)}\vec{r}_1 = \left(\frac{k_1}{k_1-1}\right)\vec{a} - \left(\frac{k_2}{k_2-1}\right)\vec{b} \quad — \quad (3)$$

$$\text{Similarly, } \frac{k_3-k_2}{(k_2-1)(k_3-1)}\vec{r}_2 = \left(\frac{k_2}{k_2-1}\right)\vec{b} - \left(\frac{k_3}{k_3-1}\right)\vec{c} \quad — \quad (4)$$

$$\text{and } \frac{k_1-k_3}{(k_3-1)(k_1-1)}\vec{r}_3 = \left(\frac{k_3}{k_3-1}\right)\vec{c} - \left(\frac{k_1}{k_1-1}\right)\vec{a} \quad — \quad (5)$$

($\vec{r}_2$ & $\vec{r}_3$ are the position vectors of P_2 & P_3 which are not shown)
(3) + (4) + (5) gives

$$\frac{(k_2-k_1)}{(k_1-1)\,(k_2-1)}\,\vec{r}_1 + \frac{(k_3-k_2)}{(k_2-1)\,(k_3-1)}\,\vec{r}_2 + \frac{(k_1-k_3)}{(k_3-1)\,(k_1-1)}\,\vec{r}_3 = 0$$

[Note : $x\,\vec{r}_1 + y\,\vec{r}_2 + z\,\vec{r}_3 = 0$]

Also sum of the co-efficient of $\vec{r}_1, \vec{r}_2$ and $\vec{r}_3$ is

$$\frac{(k_3-1)\,(k_2-k_1) + (k_1-1)\,(k_3-k_2) + (k_2-1)\,(k_1-k_3)}{(k_1-1)\,(k_2-1)\,(k_3-1)}$$

which is equal to zero .

Hence , $\vec{r}_1, \vec{r}_2$ and $\vec{r}_3$ are collinear

Shortest Distance Between Two Lines :
If two lines in space intersect at a point, then obviously the shortest distance between them is zero. Lines which do not intersect & are also not parallel are called **skew lines**. For Skew lines the direction of the shortest distance would be perpendicular to both the lines. The magnitude of the shortest distance vector would be equal to that of the projection of $\overrightarrow{AB}$ along the direction of the line of shortest distance, $\overrightarrow{LM}$ is parallel to $\vec{p} \times \vec{q}$

i.e. $\overrightarrow{LM} = \left|\text{Projection of } \overrightarrow{AB} \text{ on } \overrightarrow{LM}\right| = \left|\text{Projection of } \overrightarrow{AB} \text{ on } \vec{p}\times\vec{q}\right| = \left|\frac{\overrightarrow{AB}\,.\,(\vec{p}\times\vec{q})}{\vec{p}\times\vec{q}}\right| = \left|\frac{(\vec{b}-\vec{a})\,.\,(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right|$

1. The two lines directed along $\vec{p}$ & $\vec{q}$ will intersect only if shortest distance = 0 i.e. $(\vec{b}-\vec{a}).(\vec{p}\times\vec{q}) = 0$ i.e. $(\vec{b}-\vec{a})$ lies in the plane containing $\vec{p}$ & $\vec{q}$. $\Rightarrow \left[(\vec{b}-\vec{a})\;\vec{p}\;\vec{q}\right] = 0$.

2. If two lines are given by $\vec{r}_1 = \vec{a}_1 + K\vec{b}$ & $\vec{r}_2 = \vec{a}_2 + K\vec{b}$ i.e. they are parallel then , $d = \left|\frac{\vec{b}\times(\vec{a}_2-\vec{a}_1)}{|\vec{b}|}\right|$

16. EQUATION OF A PLANE :
Consider the plane through a given point **a**, and perpendicular to the vector **m**. If **r** is the position vector of a point P on it, **r** – **a** is parallel to the palne and therefore perpendicular to **m**. Consequently

(**r** – **a**). **m** = 0 (1)

This equation is satisfied by any point on the plane, but by no point off the plane.

Let **n** denote the unit vector perpendicular to this plane, and directed from the origin toward the plane. Then the above equationis equivalent ot (**r** – **a**). **n** = 0. The scalar product **a . n** is the resolute of $\overrightarrow{OA}$ along **n**, and is equal to the perpendicular ON from the origin to the plane. This is positive, and will be denoted by p. We may then write the equation of the plane in the form.

p – **r . n** = 0 (3)

which will be called the normal form of the equation of the plane. It should be remembered that, in this form, **n** has the sense from the origin to the plane, and p is positive.
The inclination of two planes, whose equations are

p – **r . n** = 0, p′ – **r . n**′ = 0,

is the angle θ between their normals; and this is given by

cos θ = **n . n**′.

Distance of a point from a plane. It is required to find the perpendicular distance from a point

P′(**r**′) to the plane (3), measured in the sense of **n** above. Consider the parallel plane through P′. and let p′ be the perpendicular from O to this plane. Then, as above, $p' = \mathbf{r}' . \mathbf{n}$, and the perpendicular distance from P′ to the given plane is

$$p - p' = p - \mathbf{r}' . \mathbf{n} \qquad (1)$$

This is positive for points on the same side of the plane as the origin, negative for points on the opposite sides.

To find the distance from P′ to the plane, measured i the direction of the unit vector **b**, let a parallel to **b** drawn through P′ cut the plane in H. Then if d is the length of P′H, the position vector of H is $\mathbf{r}' + d\mathbf{b}$; and, since this point lies on the given plane, we have

$$p - (\mathbf{r}' + d\mathbf{b}) . \mathbf{n} = 0,$$

so that $$d = (p - \mathbf{r}' . \mathbf{n}) / \ \mathbf{b} . \mathbf{n}.$$

We find the equations of the planes which bisect the angles between two given planes

$$p - \mathbf{r} . \mathbf{n} = 0, \qquad p' - \mathbf{r} . \mathbf{n}' = 0.$$

These follows from the fact that a point on either bisector is equidistant from the two planes. For points on the plane bisecting the angle in which the origin lies, the two perpendicular distance have the same signs. but for points n the other bisector, opposite signs. The equation of the former bisector is therefore

$$p - \mathbf{r} . \mathbf{n} = p' - \mathbf{r} . \mathbf{n}',$$

or $$p - p' = \mathbf{r} . (\mathbf{n} - \mathbf{n}')$$

and that of the other bisector is

$$p + p' = \mathbf{r} . (\mathbf{n} + \mathbf{n}')$$

Angle between the 2 planes is the angle between 2 normals drawn to the planes and the angle between a line and a plane is the compliment of the angle between the line and the normal to the plane.

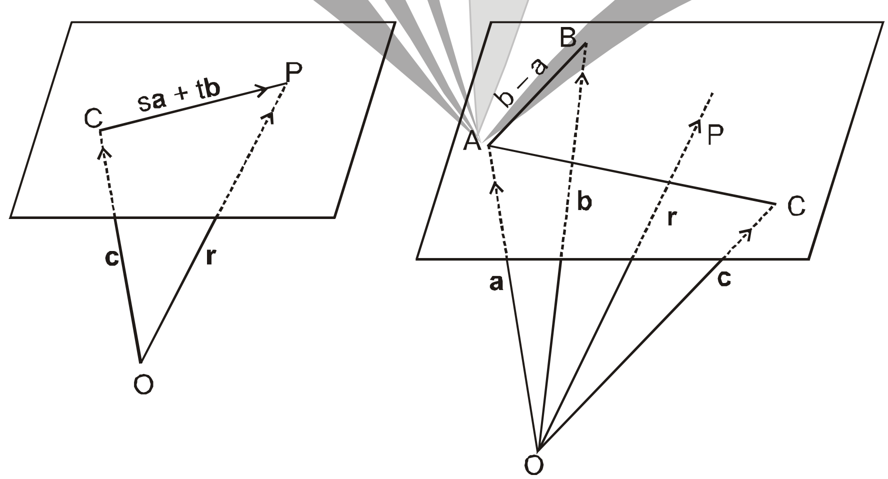
Parametric equation of plane.

Ex.47 To find a vector equatio of the plane through the point C parallel to **a**b and **b**.

Sol. Let **c** be the position vector of C, and **r** that of any point P on the given plane. The vector $\overrightarrow{CP}$ is coplanar with **a** and **b**, and may therefore be expressed as $s\mathbf{a} + t\mathbf{b}$ as in (1). Then

$$\mathbf{r} = \overrightarrow{OP} = \overrightarrow{OC} + \overrightarrow{CP}$$

$$= \mathbf{c} + s\mathbf{a} + t\mathbf{b}. \qquad (2)$$



(a) (b)

This is the required equation to the plane, the numbers s, t varying as P moves over the plane. And no point off the plane can be represented by (2).

To find the equation of the plane through the three points A, B, C whose position vectors are **a, b, c** we observe that $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ and $\overrightarrow{AC} = \mathbf{c} -$;

so that the plane is one through A parallel to $\mathbf{b} - \mathbf{a}$ and $\mathbf{c} - \mathbf{a}$. Its equation is therefore

$$\mathbf{r} = \mathbf{a} + s(\mathbf{b} + \mathbf{a}) + t(\mathbf{c} - \mathbf{a})$$

or $$\mathbf{r} = (1 - s - t)\mathbf{a} + s\mathbf{b} + t\mathbf{c}. \qquad (3)$$

The equation (1), (2), (3) involve each two variable numbers s, t.

The four points A, B, C, P in fig. (b) are coplanar; and if linear relation (3) connecting their position vectors is written $(1 - s - t)\mathbf{a} + s\mathbf{b} + t\mathbf{c} - \mathbf{r} = \mathbf{0}$,

with all the terms on one side, the algebraic sum of the coeffcients of the vectors is zero. This is the necessary and sufficient condition that four points should be coplanar.

Ex.48 If any point O within or without a tetrahedron ABCD is joined to the vertices, and AO, BO, CO, DO are produced to cut the planes of the opposite faces in P, Q, R, S respectively, then $\Sigma \frac{OP}{AP} = 1$.

Sol. With O as origin let the position vectors of A, B, C, D be **a, b, c, d** respectively. Any one of these vectors may be expressed in terms of the other three, so that there is a linear relation connecting them which may be written

$$l\mathbf{a} + m\mathbf{b} + n\mathbf{c} + p\mathbf{d} = \mathbf{0}. \qquad (1)$$

The equation of the line AP is $\mathbf{r} = -u\mathbf{a}$, u being positive for points of theline which lie on the oposite side of the origin from A. In virtue of (1) we may write this equation as

$$l\mathbf{r} - u(m\mathbf{b} + n\mathbf{c} + p\mathbf{d}) = \mathbf{0} \qquad (2)$$

For the particular point P, in which this line cuts the plane BCD, the four points **r, b, c, d** are coplanar. Hence the sum of the coefficients in (2) is zero, so that $u = l/(m + n + p)$, and therefore

$$\frac{OP}{AP} = \frac{u}{1+u} = \frac{l}{l+m+n+p}.$$

the other ratio may be written down by cyclic permutation of the symbols, and their sum is obviously equal to unity.

Ex.49 Find the intersection of the line joining the points (1, – 2, –1) and (2, 3, 1) with the plane through the points (2, 1, –3), (4, –1, 2) and (3, 0, 1).

Sol. The equation of the straight line is

$$(x, y, z) = (1, -2, 1-1) + t(1, 5, 2)$$

and that of the plane is

$$(x, y, z) = (2, 1, -3) + u(2, -2, 5) + v(1, -1, 4).$$

For the point of intersection the two equation give the same values of x, y, z. Hence on equating corresponding components in the two expressions for (x, y, z) we find

$$1 + t = 2 + 2u + v, \qquad -2 + 5t = 1 - 2u - v,$$

and a third equation. From the first two by addition we find t = 2/3, so that the point of intersection is (5, 4, 1)/3.

Ceva's Theorem. If the lines joining the vertices A, B, C of a triangle to point P, in the plane of the triangle, cut the opposite sides in D, E, F respectively, then prove that the product of the ratios in which these points divide BC, CA, AB is equal to unity.

Since A, B, C, P are coplanar, their position vectors satisfy a linear relation of the form

$$l\mathbf{a} + m\mathbf{b} + n\mathbf{c} + h\mathbf{p} = \mathbf{0}, \qquad l + m + n + h = 0,$$

the four coefficients being different from zero. From these equations it follows that

$$\frac{m\mathbf{b} + n\mathbf{c}}{m+n} = \frac{l\mathbf{a} + h\mathbf{p}}{l+h} = \mathbf{d}.$$

The first of these expressions is the position vector of a point dividing BC in the ratio n : m, and the second is that of a poitn dividing AP in the ratio h : l. Each therefore represents that of the point D in which the line AP intersects BC. Thus BD : DC = n : m. Similarly E and F divide CA and AB in the ratios l : n and m : l respectively. The product of these three ratios is unity.

Planes satisfying various condtions. We shall now see how the triple products may be used in finding the equation of a plane subject to certain conditions. Let us examine the following typical cases

(i) **Plane through three given points** A, B, C. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be the position vectors of the points relative to an assigned origin O, and $\mathbf{r}$ that of a variable point P on the plane. Since P, A, B, C all lie on the palne, the vectors $\mathbf{r} - \mathbf{a}$, $\mathbf{a} - \mathbf{b}$, $\mathbf{b} - \mathbf{c}$ are coplanar, and their scalar triple product is zero. Hence

$$(\mathbf{r} - \mathbf{a})\ (\mathbf{a} - \mathbf{b}) \times (\mathbf{b} - \mathbf{c}) = 0.$$

If two expand this, and neglect the triple products in which any vector occurs twice, the equation becomes

$$\mathbf{r}\ .\ (\mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} + \mathbf{a} \times \mathbf{b}) = [\mathbf{abc}].$$

Thus the plane is perpendicular to the vector area of the triangle ABC.

(ii) **Plane through a given point parallel to two given straight lines.** Let $\mathbf{a}$ be the given point, and $\mathbf{b}, \mathbf{c}$ two vectors parallel to the given lines. Then $\mathbf{b} \times \mathbf{c}$ is perpendicular to the plane; and we have only to write down the equation of the plane through $\mathbf{a}$ perpendicular to $\mathbf{b} \times \mathbf{c}$. By Art. 24 this is

$$(\mathbf{r} - \mathbf{a}).\ \mathbf{b} \times \mathbf{c} = 0,$$

that is $\quad \mathbf{r}\ .\ \mathbf{b} \times \mathbf{c} = [\mathbf{abc}].$

(iii) **Plane containing a given straight line and parallel to another.** Let the first line be represented by $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, while the second is parallel to $\mathbf{c}$. Then the plane in question contains the point $\mathbf{a}$, and is parallel to $\mathbf{b}$ and $\mathbf{c}$. Its equation is therefore, by the last case,

$$\mathbf{r}\ .\ \mathbf{b} \times \mathbf{c} = [\mathbf{abc}].$$

(iv) **Plane through two given points and parallel to a given straight line.** Let $\mathbf{a}, \mathbf{b}$ be two given points, and $\mathbf{c}$ a vector parallel to the given straight line. The required plane then passes thrugh $\mathbf{a}$ and is parallel to $\mathbf{b} - \mathbf{a}$ and $\mathbf{c}$. Its equation is therefore

$$\mathbf{r}\ .\ (\mathbf{b} - \mathbf{a}) \times \mathbf{c} = [\mathbf{a}, \mathbf{b} - \mathbf{a}, \mathbf{c}]$$
$$= [\mathbf{abc}].$$

(v) **Plane containing a given straight line and a given point.** Let $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ be the given straight line and c the given point. Then the plane in question passes throught the two points $\mathbf{a}, \mathbf{c}$ and is parallel to $\mathbf{b}$. Hence by (iv) is its equation is

$$\mathbf{r}\ .\ (\mathbf{a} - \mathbf{c}) \times \mathbf{b} = [\mathbf{abc}].$$

Condition of intersection of two straight lines. Let the equations of the given straight lines be

$$\mathbf{r} = \mathbf{a} + t\mathbf{b}, \qquad \mathbf{r} = \mathbf{a}' + s\mathbf{b}',$$

so that they pass through the points $\mathbf{a}, \mathbf{a}'$ and are parallel to $\mathbf{b}, \mathbf{b}'$ respectively. If they intersect, their common palne must be parallel to each of the vectors $\mathbf{b}, \mathbf{b}', \mathbf{a} - \mathbf{a}'$, whose scalar triple product is therefore zero. Hence the required condition is

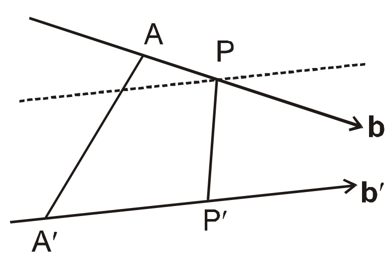
$$[\mathbf{b}, \mathbf{b}', \mathbf{a} - \mathbf{a}'] = 0$$

The common perpendicular to two skew lines. Let the equations of the two straight lines be

$$\mathbf{r} = \mathbf{a} + t\mathbf{b}, \qquad \mathbf{r} = \mathbf{a}' + s\mathbf{b}'.$$

Then the vector $\mathbf{n} = \mathbf{b} \times \mathbf{b}'$ is perpendicular to both lines, and therefore parallel to their common perpendicular P′P. If A, A′ are the points $\mathbf{a}, \mathbf{a}'$ respectively, the length p of this common perpendicular is equal to the length of the projection of A′A on $\mathbf{n}$.

Hence $\quad p = \dfrac{\mathbf{n}.(\mathbf{a} - \mathbf{a}')}{\mathbf{n}} = \dfrac{1}{n}\ [\mathbf{b}, \mathbf{b}', \mathbf{a} - \mathbf{a}'].$



The equation of the plane containing the first line and the common perpendicular to the two lines is $[\mathbf{r} - \mathbf{a}', \mathbf{b}', \mathbf{b} \times \mathbf{b}'] = 0$. The point P′ is that in which the second line meets this plane. Similarly the equation of the plane containing the second line and thecommon perpendcular is **[r – a′, b′, b × b′] = 0.**These two planes determine the line P′P.

Ex.50 Find the shortest distance between the straight lines through the poins A (6, 2, 2) and A′(–4, 0, –1) in the direction (1, –2, 2) and (3, –2, –2) respectively. Also find the feet, P and P′, of the commn perpendicular.

Sol. In this case **b** × b′ = 4(2, 2, 1), and the unit vector in this direction is (2, 2, 1)/3. The shortest distance is the projection of A′A on this direction, so that

$$p = (10, 2, 3). (2, 2, 1)/3 = 27/3 = 9.$$

The pair of skew lines is threfore right-handed. The equation of the plane APP′ is 2x – y – 2z = 6; and the second line meets this in the point P′ (–1, –2, –3). Shown similarly that P is the point (5, 4, 0).

Also show that the moment about either line, of a unit vector localized in the other, is 36/√17.

Ex.51 Examine similarly the pair of line determined by the equations

$$3x – 4y – z + 5 = 0 = 3x – 6y – 2z + 13$$

and $$3x + 4y\ \ 3z + 2 = 0 = 3x – 2y +6z + 17.$$

Sol. The first line, being the intersection of two planes, is perpendicular to both normals, and therefore has the direction of the vector **b** = (2, 3, –6). One point on the line is A(5, 6, – 4). Similarly the second line has the direction of **b**′ = (2, –3, –2), and one point on the line is A′(1, – 5, –5). These directions along the lines are chosen because they are inclined at an acute angle. Then **b** × **b′** = –4(6, 2, 3), and the unit vector in this direction is –(6, 2, 3)/7. The projection of A′A on this direction is then

$$p = –(6, 2, 3). (4, 11, 1)/7 = –49/7 = –7.$$

The pair of liens is threfore left-handed. The equation of the plane containing the second lines and the common perpendicular is

$$5s + 18y – 22z = 25;$$

and this plane is cut by the first line in P(3, 3, 2). Similarly find P′(–3, 1, –1). Also show that the moment about either line, of a unit vector localized in the other, is –28/√17.

Ex.52 Find the equation of the plane through the line **d, m** parallel to **c**.

Sol. Since the required plane is paeallel to **d** and **c**, its normal is parallel to **d** × **c**. If **b** is a point on the given line, then **b** × **d** = **m**,

and, **r** being a current point on the plane, **r** – **b** is parallel to the plane, and theregore perpendicular to the normal. Hence (**r** – **b**). (**d** × **c**) = 0,

which may be written [**rdc**] = **m . c**.

This is the required equation of the plane.

Ex.53 Find the equatio of the plane through the point **a** and the line **d, m**.

Sol. If **b** is a point on the given line, then **b** × **d** = **m**. The plane is parallel to **b** – **a** and **d**, and its normal is parallel to (**b** – **a**) × **d**. Hence if **r** is a current point on the plane (**r** – **a**) is also parallel to the plane, and therefore (**r** – **a**). (**b** – **a**) × **d** = 0,

which may be written **r . m** – [**rad**] = **a . m**.

17. SPHERE

Equation of sphere. Consider the sphere of centre C and radius a. If p is any point on the surface of the sphere and **r, c** are the position vectors of P, C respectively relative to an origin O, the vector $\overrightarrow{CP}$ = **r** – **c** has a length equal to the radius, and therefore $(\mathbf{r} – \mathbf{c})^2 = a^2$.

If we put $k = c^2 – a^2$ we may write this relation

$$\mathbf{r}^2 – 2\mathbf{r} . \mathbf{c} + k = 0. \qquad (1)$$

This equation is satisfied by the position vector of every point on the surface of the sphere, and by no other. It will therefore be called the equation of the sphere relative to the otigin O.

The first member of (1), regarded as a function of the vector **r**, may for conveience be denoted by F(**r**); and the equation written briefly F(**r**) = 0.

Consider the points of intersection of the surface with the straight line through the point D parallel to the unit vector **b**. The equation of this line is

$$\mathbf{r} = \mathbf{d} + t\mathbf{b}, \qquad (2)$$

where **d** is the position vector of D. The values of **r** for the points of intersection satisfy both (1) and (2). If then we eliminate **r** from these equations we find, for the values of t correcponding to the points of intersection, the quadratic equation

$$t^2 + 2\mathbf{b} \cdot (\mathbf{d} - \mathbf{c})t + (\mathbf{d}^2 - 2\mathbf{c} \cdot \mathbf{d} + k) = 0, \qquad (3)$$

the coefficient of t^2 being unity, since **b** is a unit vector. The equation has two roots, t_1, t_2 whcih are real if $\{\mathbf{b} \cdot (\mathbf{d} - \mathbf{c})\}^2 \geq F(\mathbf{d})$.

Corresponding to these two roots there are two points of intersection, P, Q, such that $DP = t_1$ and $DQ = t_2$. The product of these rotos is equal to the absolute term of (3), i.e. DP . DQ = F(**d**).

This is independent of **b**, and is therefore the same for all straight lines through D. If the points P, Q tend to coincidence at T, the straight line becomes a tangent, and we have

$$DT^2 = DP \,.\, DQ = F(\mathbf{d}). \qquad (4)$$

Thus F(**d**) measures the square on the tangent from D to the surface of the sphere. It is called the power of the point D with respect to the sphere, and it is equal to $CD^2 - a^2$. In particular F(0) = k is the power of the origin, O. If O is within the sphere k is negative, and the tangents from O are imaginary. The tangents from D are generators of a cone, called the tangent cone, having its vertex at D and enveloping the sphere.

Equation of the tangent plane at a point. If D is a point on the surface of the sphere, F(**d**) = 0, and one root of the equation (3) is zero. In order that the line (2) should touch the surface, the other root also must be zero, for a tangent line intersects the sufrace in two coincident points. If then both roots of (3) are zero, $\mathbf{b} \cdot (\mathbf{d} - \mathbf{c}) = 0$,

showing that the tangent line is perpendicular to the radius CD. If **r** is any point on the tangent line, this condition is equivalent to

$$(\mathbf{r} - \mathbf{d}) \cdot (\mathbf{d} - \mathbf{c}) = 0 \qquad (5)$$

This equation represents a plane through D perpendicular to CD; showing that all tangent lines through D lie on this plane, whcih is called the tangent plane at D. Adding the zero quantity F(**d**) to the first member of (5), we may write the equation

$$\mathbf{r} \cdot \mathbf{d} - \mathbf{c} \cdot (\mathbf{r} \cdot \mathbf{d}) + k = 0, \qquad (6)$$

which we shall take as the standard form of the equation of the tangent plane at the poing **d**.

Since a tangent plane is perpendicular to the radius to the point of contact, the square of the perpendicular from the centre C to a tangent plane must be equal to a^2. Hence the condition that the plane $p - \mathbf{r} \cdot \mathbf{n} = 0$ should touch the sphere (1) is taht

$$(p - \mathbf{c} \cdot \mathbf{n})^2 = c^2 - k. \qquad (7)$$

Further if two spheres cut each other at right angles, the tangent plane to either at a point of intersection passes through the centre of the other. Hence the square on the line joining their centres is equal to the sum of the squares on their radii. If then the equations of the two spheres are $\mathbf{r}^2 - 2\mathbf{r} \cdot \mathbf{c} + k = 0$, $\mathbf{r}^2 - 2\mathbf{r} \cdot \mathbf{c}' + k' = 0$,

the condition that they should cut ortogonally is

$$(\mathbf{c} - \mathbf{c}')^2 = a^2 + a'^2 = c^2 - k + c'^2 - k',$$

that is

$$2\mathbf{c} \cdot \mathbf{c}' - (k + k') = 0. \qquad (8)$$

18. APPLICATION OF VECTORS :

Work done by a force. A force acting on a particle does words when the particle is dixplaced in a directio which is not perpendicular to the force. The work doen is a scalar quantity jointly proportional to the force and the resolved part of the displacement in the direction of the force. We choose the unti quantity of work as that done when a particle, acted on by unit force, is displaced unit distance in the direction of the force. Hence, if **F**, **d** are vectors representing the force and the displacement respectivly, inclined at an angle θ, the measure of the work done is

$$Fd \cos \theta = \mathbf{F} \cdot \mathbf{d}.$$

The work done is zero only when **d** is perpendiclar to **F**.

Suppose next that the particle is acted on by several force $\mathbf{F}_1, \mathbf{F}_2, ..., \mathbf{F}_n$. Then during a displacement

d of the particle the separate forces do quantities of work $\mathbf{F}_1 \,.\, \mathbf{d}$, $\mathbf{F}_2 \,.\, \mathbf{d}$,..., $\mathbf{F}_n \,.\, \mathbf{d}$. The total work done is $\sum_{1}^{n} \mathbf{F} \,.\, \mathbf{d} = \mathbf{d} \,.\, \Sigma \mathbf{F} = \mathbf{d} \,.\, \mathbf{R}$,

and is therefore the same as if the system of forces were replaced by its resulatant **R**.

Note. A force represented by the vector **F** may be conveniently referred to as a force **F**. No misunderstanding is possible, for our Clarendon symbols always denote length-vectors. Similarly we may speak of a displacement **d**, or a velocity **v**, as we have already done of a point **r**.

Moment of force about a point. A force is an example of a vector quantity localized in a straight line. The single vector **F**, used above to represent the force, does so only in magnitude and direction. To specify the lien of action another vector is necessary along with **F**. The most useful for this purpose is the moment of the force about a specified point. Let O be any covenient point, and **r** the position vector relative to **O** of any point P on the line of action of the force. Then the moment of the force about the point O is defined as the vector $\mathbf{m} = \mathbf{r} \times \mathbf{F}$.

This vector, also called the torque of the force about O, is perpendicular to the plane of **r** and **F**, and therefore to the plane containng O and the line of action of the force. Its magnitude is pF, where p is the length of the perpendicular ON to the line of action. Conversely, given **F** and **m**, the force is specified in magnitude, direction, and line of action. For the line of action lies in the plane through O perpendicular to **m**, which is the plane OPN. Its direction is that of **F**, and its distance p from O is such that pF = m. It lies on that side of O which makes a rotation from OP to **F** positive with respect to the direction of **m**.

If there are several forces $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$..... acting through the same point P they have a resultant $\mathbf{R} = \Sigma \mathbf{F}$. The moment of this resultant about O is

$$\mathbf{r} \times \mathbf{R} = \mathbf{r}\,(\mathbf{F}_1 + \mathbf{F}_2 + \ldots.)$$
$$= \mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2 + \ldots.,$$

and therefore equal to the vector sum of the moments of the separate forces, Thus, if the system of forces through P is replaced by its resultant, the moment about any poit remains unchanged.

Express the vectors **F**, **r** in terms of their rectangular components, as

$$\mathbf{F} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}, \qquad \mathbf{r} = x\mathbf{i} + y\mathbf{i} + z\mathbf{k}.$$

Then the moment of the force **F** about O is

$$\mathbf{m} = (yZ - zY)\mathbf{i} + (zX - xZ)\mathbf{j} + (xY - yX)\mathbf{k}.$$

In this expression the coefficients of **i, j, k** are the moments of the force about the coordinate axes. And as the system of rectangular axes through O may be chosen so that one of them has any assigned direction, it follows that the moment of a force **F** about any straight line through O is the resolved part along this line of the moment of **F** about O. Whit the above notation the moments about the coordinate axes are $\mathbf{m} \,.\, \mathbf{i}$, $\mathbf{m} \,.\, \mathbf{j}$ and $\mathbf{m} \,.\, \mathbf{k}$ respectively. Thus the moment of a force about a point is a vector, while the moment about a straight line is a scalar quantity.

If there are seveal concurrent force it follows from the above that the moment of the resultant about any axis through O is equal to the sum of the moments of the several forces about that axis.

Angular vecloty of a rigid body about a fixed axis. Consider the motion of a rigid body rotating about a fixed axis ON at the rate of ω radians per second. It will be shown in Art. 86 that, if one pont O of the body is fixed, the instantaneous motion of the body is one of rotation about such an axis thrugh O, every point on the axis being instantaneously at rest. For the present we take the rotation as given about the axis ON. The angular velocity of the body is uniquely specified by a vector A whose modulus is ω and whoe direction is parallel to the axis, and in the positive sense relative to the rotation.

Let O be any point on the fixed axis, P a point fixed in the body, **r** the position vector P relative to O, and PN perpendicular to the axis of rotation. Then the particle at P is moving in a circular path, with centre N and radius p =PN. Its velocity is therefore perpendicular to the plane OPN and of magnitude $p\omega = r\omega \sin PON$. Such a velocity is represented by the vector $\mathbf{A} \times \mathbf{r}$, the sense of this vector being the same as that of the velocity. In other words, the velocity of the particle at P is $\mathbf{v} = \mathbf{A} \times \mathbf{r}$.

Note :

(a) Work done against a constant force $\vec{F}$ over a displacement $\vec{s}$ is defined as $\vec{W} = \vec{F}.\vec{s}$

(b) The tangential velocity $\vec{V}$ of a body moving in a circle is given by $\vec{V} = \vec{w} \times \vec{r}$ where $\vec{r}$ is the pv of the point P .

(c) The moment of $\vec{F}$ about 'O' is defined as $\vec{M} = \vec{r} \times \vec{F}$ where $\vec{r}$ is the pv of P wrt 'O'. The direction of $\vec{M}$ is along the normal to the plane OPN such that $\vec{r}, \vec{F}$ & $\vec{M}$ form a right handed system .

(d) Moment of the couple = $(\vec{r}_1 - \vec{r}_2) \times \vec{F}$ where $\vec{r}_1$ & $\vec{r}_2$ are pv's of the point of the application of the forces $\vec{F}$ & $-\vec{F}$.