VKR Classes

VKR Sir
B.Tech., IIT DELHI
with you since 15 years

JEE Advanced

Time : 2 hr. Test Paper 09 Date 25/01/15 Batch - R Marks : 120

SINGLE CORRECT CHOICE TYPE [3, –1]

1. From the extremities P and Q of a focal chord of a parabola, perpendiculars PM and QL are drawn on the axis of the parabola. If S is its focus and V, its vertex, then VM, VS and VL are in

(A) A.P. (B) G.P. (C) H.P. (D) None of these

2. Let Z is complex satisfying the equation $z^2 - (3 + i)z + m + 2i = 0$, where $m \in R$. Suppose the equation has a real root. The additive inverse of non real root, is

(A) 1 – i (B) 1 + i (C) – 1 – i (D) –2

3. Given three points P_1, P_2 and P_3 with position vectors $\vec{r}_1 = 2\hat{i} - \hat{j} + \hat{k}$; $\vec{r}_2 = \hat{i} + 3\hat{j}$ and $\vec{r}_3 = \hat{i} + \hat{j} - 3\hat{k}$ respectively. Equation of a line orthogonal to the plane containing these points and passing through the centroid P_0 of the triangle $P_1P_2P_3$, is

(A) $\vec{r} = \frac{1}{3}\left(4\hat{i} + 3\hat{j} - 2\hat{k}\right) + \lambda(14\hat{i} + 3\hat{j} + 2\hat{k})$ (B) $\vec{r} = \frac{4}{3}\hat{i} + \hat{j} - \frac{2}{3}\hat{k} + \lambda(14\hat{i} + 3\hat{j} - 2\hat{k})$

(C) $\vec{r} = 4\hat{i} + 3\hat{j} - 2\hat{k} + \lambda(14\hat{i} + 3\hat{j} - 2\hat{k})$ (D) $\vec{r} = \frac{1}{3}\left(4\hat{i} + 3\hat{j} - 2\hat{k}\right) + \lambda(10\hat{i} + 3\hat{j} - 2\hat{k})$

4. TP & TQ are tangents to the parabola, $y^2 = 4ax$ at P & Q. If the chord PQ passes through the fixed point (– a, b) then the locus of T is-

(A) ay = 2b (x – b) (B) bx = 2a (y – a) (C) by = 2a (x – a) (D) ax = 2b (y – b)

5. The locus of the centre of a circle, which intercepts a chord of given length 2a on the axis of x and passes through a given point on the axis of y distant b from the origin, is the curve

(A) $x^2 - 2yb + b^2 = a^2$ (B) $x^2 + 2yb + b^2 = a^2$ (C) $y^2 - 2xb + b^2 = a^2$ (D) none of these

6. If there are three square matrix A, B, C of same order satisfying the equation $A^2 = A^{-1}$ and let B = A^{2^n} & C = $A^{2^{(n-2)}}$ then which of the following statements are true?

(A) det. (B – C) = 0 (B) (B + C)(B – C) = 0 (C) B must be equal to C (D) none

7. The three vectors $\hat{i} + \hat{j}$, $\hat{j} + \hat{k}$, $\hat{k} + \hat{i}$ taken two at a time form three planes. The three unit vectors drawn perpendicular to these three planes form a parallelopiped of volume :

(A) $\frac{1}{3}$ (B) 4 (C) $\frac{3\sqrt{3}}{4}$ (D) $\frac{4}{3\sqrt{3}}$

8. $\lim_{n\to\infty} n \int_0^{\pi/2} \left(1 - \sqrt[n]{\sin x}\right)dx$ equals

(A) $\pi \ln 2$ (B) $\frac{\pi}{2} \ln 2$ (C) $-\frac{\pi}{2} \ln 2$ (D) none

COMPREHENSION [3, -1]

Comprehension # 1

Let $A(z_1)$ be the point of intersection of curves $\arg(z-2+i) = \frac{3\pi}{4}$ and $\arg(z+i\sqrt{3}) = \frac{\pi}{3}$. $B(z_2)$ be the point on the curve $\arg(z+i\sqrt{3}) = \frac{\pi}{3}$ such that $|z_2 - 5|$ is minimum and $C(z_3)$ be the centre of circle $|z-5| = 3$.

9. The area of triangle ABC is equal to

(A) $4\sqrt{3}$ (B) $\frac{3\sqrt{3}}{2}$ (C) $2\sqrt{3}$ (D) 4

10. The equation of straight line passing through origin and perpendicular to line joining $A(z_1)$ and $B(z_2)$ on the complex plane is equal to

(A) $z = \lambda(2+i\sqrt{3})$ (B) $z = \lambda(-\sqrt{3}+i)$ (C) $z = \lambda(1+i\sqrt{3})$ (D) $z = \lambda(\sqrt{3}+i)$

Comprehension # 2

A tangent is drawn to the parabola $y^2 = 4x$ at the point P which is the upper end of latus rectum.

11. Image of the parabola $y^2 = 4x$ in the tangent line at the point P is

(A) $(x+4)^2 = 16y$ (B) $(x+2)^2 = 8(y-2)$ (C) $(x+1)^2 = 4(y-1)$ (D) $(x-2)^2 = 2(y-2)$

12. Area enclosed by the tangent line at P, x-axis and the parabola is

(A) $\frac{2}{3}$ (B) $\frac{4}{3}$. (C) $\frac{14}{3}$ (D) none

Comprehension # 3

Let C be a curve defined by $y = e^{a+bx^2}$. The curve C passes through the point (P(1, 1) and the slope of the tangent at P is (–2). Also C_1 and C_2 are the circles $(x-a)^2 + (y-b)^2 = 3$, $(x-6)^2 + (y-11)^2 = 27$ respectively.

13. The length of the shortest line segment AB which is tangent to C_1 at A and to C_2 at B is

(A) $9\sqrt{3}$ (B) $10\sqrt{3}$ (C) 11 (D) 12

14. If f is a real valued derivable function satisfying $f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)}$ with $f'(1) = 2$. Then the value of the integral $\int_b^a f(x)\, d(\ell n\, x)$ is equal to

(A) 0 (B) $\frac{e^2 - e^{-2}}{2}$ (C) $\frac{e^{-2} - e^2}{2}$ (D) 2

Comprehension # 4

Consider $N = \begin{vmatrix} 77 & 44 \\ 7 & 30 \end{vmatrix} = \prod_{i=1}^{4} (a_i)^{b_i}$ where b_i's $\in$ N and a_i's are prime numbers and $a_i < a_{i+1}$ $\forall$ i. Let two circles be defined as

$C_1 : [x^2 \;\; y^2] \begin{bmatrix} a_1 \\ b_1 + b_2 \end{bmatrix} = [8b_1]$ and $C_2 : [(x+y)^2 \;\; (x-y)^2] \begin{bmatrix} b_3 \\ b_4 \end{bmatrix} = [2b_2]$

15. Radius of the circle which passes through centre of the circle C_2 and touches the circle C_1, is

(A) 1 (B) $\frac{1}{2}$ (C) $\frac{\sqrt{3}}{2}$ (D) $\frac{1}{\sqrt{2}}$

16. If the line $x + \lambda y = 1 + k\lambda$ always passes through a fixed point on the circle C_1 for all values of λ then the length of tangent from the point $(1, \sqrt{3}k)$ to the circle C_2, is

(A) k (B) k^2 (C) 2k (D) $2k^2$

MULTIPLE CORRECT CHOICE TYPE [3, 0]

17. Let f : [0, 1] → R be a continuous function such that x f(y) + yf(y) ≤ 1 for any x, y in the domain. Let $I = \int_0^1 f(x)\,dx$ then

(A) $I = \int_0^{\pi/2} f(\sin\theta)\cos\theta\, d\theta$ (B) $I = \int_0^{\pi/2} f(\cos\theta)\cos\theta\, d\theta$

(C) $I \le \frac{\pi}{4}$ (D) $I \le \frac{\pi}{8}$

18. Which of the following statement(s) is/are correct?

(A) Let f and g be defined on R and c be any real number. If $\lim_{x\to c} f(x) = b$ and g (x) is continuous at x = b then $\lim_{x\to c} g(f(x))$ = g (b).

(B) There exist a function f : [0, 1] → R which is discontinuous at every point in [0, 1] and $|f(x)|$ is continuous at every point in [0, 1].

(C) If f (x) and g (x) are two continuous function defined from R → R such that f (r) = g (r) for all rational numbers 'r' then f (x) = g (x) ∀ x ∈ R.

(D) If f (a) and f (b) possesses opposite signs then there must exist atleast one solution of the equation f (x) = 0 in (a, b) provided f is continuous in [a, b].

19. The equation $\tan^{-1}x = ax^2 + b$ has two solutions when

(A) $a = 1, b = -\frac{8}{5}$ (B) a = 2, b = 0 (C) $a = \frac{1}{4}, b = \frac{\pi-1}{4}$ (D) $a = \frac{1}{4}, b = \frac{\pi-2}{4}$

20. If $f(x) = x^3 - x^2 + 100x + 2002$, then

(A) f (1000) > f (1001) (B) $f\left(\frac{1}{2000}\right) > f\left(\frac{1}{2001}\right)$

(C) f (x – 1) > f (x – 2) (D) f (2x – 3) > f (2x)

21. Let $\vec{b}$ and $\vec{c}$ are non-collinear vectors. If $\vec{a}(\vec{b} \times \vec{c}) + (\vec{a}\,.\,\vec{b})\vec{b} = (4 - 2x - \sin y)\vec{b} + (x^2 - 1)\vec{c}$ and $(\vec{c}\,.\,\vec{c})\vec{a} = \vec{c}$, then

(A) x = 1 (B) x = –1 (C) $y = (4n + 1)\frac{\pi}{2}$, n ∈ 1 (D) $y = (2n + 1)\frac{\pi}{2}$, n ∈ 1

22. If Z is a non-real complex number, then the possible values of $\frac{\operatorname{Im} Z^5}{\operatorname{Im}^5 Z}$, are

(A) – 1 (B) – 2 (C) – 4 (D) – 5

23. The extremum of the function, $f(x) = |x^2 + 2x - 3| + (3/2)\ln x$, x ∈ [1/2, 4] occur at :

(A) x = 1 (B) x = 3 (C) x = 1/2 (D) x = 4

24. In which of the following cases the given equations has atleast one root in the indicated interval ?

(A) x – cos x = 0 in (0, π/2)

(B) x + sin x = 1 in (0, π/6)

(C) $\frac{a}{x-1} + \frac{b}{x-3} = 0$, a, b > 0 in (1, 3)

(D) f(x) – g(x) = 0 in (a, b) where f and g are continuous on [a, b] and f(a) > g(a) and f(b) < g(b).

25. The equation of the curve passing through (3 , 4) & satisfying the differential equation,

$y\left(\frac{dy}{dx}\right)^2 + (x-y)\frac{dy}{dx} - x = 0$ can be

(A) $x - y + 1 = 0$ (B) $x^2 + y^2 = 25$

(C) $x^2 + y^2 - 5x - 10 = 0$ (D) $x + y - 7 = 0$

26. The inequality $\sqrt{x^{\log_2 \sqrt{x}}} \geq 2$ is satisfied by

(A) only one value of x (B) $x \in \left(0, \frac{1}{4}\right]$ (C) $x \in [4, \infty)$ (D) $1 < x < 2$

INTEGER ANSWER TYPE **[3, 0]**

27. The normal chord at a point 't' on the parabola $y^2 = 4ax$ subtends a right angle at the vertex . Then t^2 is equal to

28. A movable parabola touches the x and the y-axes at (1, 0) and (0, 1). The locus of the focus of the parabola is $ax^2 - ax + ay^2 - ay + 1 = 0$ where a is

29. If $f(x) = a\,|\cos x| + b\,|\sin x|$ $(a, b \in R)$ has a local minimum at $x = -\frac{\pi}{3}$ and satisfies $\int_{-\pi/2}^{\pi/2} (f(x))^2\, dx = 2$.

Find the values of a and b and hence find b^2/a^2 .

30. Let a and b be real numbers greater than 1 for which there exists a positive real number c, different from 1, such that $2(\log_a c + \log_b c) = 9 \log_{ab} c$. The largest possible value of $\log_a b$ is

31. The solutions of the equation, $\left(4\sqrt{\cos\frac{x}{2}} - 5 - \frac{\sqrt{2}}{2}\right)^2 + \sqrt{2}\left(4\sqrt{\cos\left(\frac{x}{2}\right)} - 5 - \frac{\sqrt{2}}{2}\right) - \frac{\cos x}{2} = 0$ form an A.P. with common difference $k\pi/2$, where k is

32. If $0 \leq [x] < 2$, $-1 \leq [y] < 1$ and $1 \leq [z] < 3$ where [·] denotes greatest integral function then the maximum value of the determinant

$$D = \begin{vmatrix} [x]+1 & [y] & [z] \\ [x] & [y]+1 & [z] \\ [x] & [y] & [z]+1 \end{vmatrix} \text{ is}$$

33. Let C be the curve passing through the point (1, 1) has the property that the perpendicular distance of the origin from the normal at any point P of the curve is equal to the distance of P from the x–axis. If the area bounded by the curve C and x-axis in the first quadrant is $\frac{k\pi}{2}$ square units, then find the value of k.

34. If ω is an imaginary cube root of unity, then the value of $(p + q)^3 + (p\omega + q\omega^2)^3 + (p\omega^2 + q\omega)^3$ is $k(p^3 + q^3)$, where k is

35. Two points A, B whose position vectors are $3\hat{i} + \hat{j} - 2\hat{k}$ and $\hat{i} - 3\hat{j} - \hat{k}$. One point P divide segment AB in ratio (a – 1) : (a + 1) internally and another point Q divide AB externally in ratio (a + 4) : a. If the length of the segment PQ comes out to be $\frac{5\sqrt{21}}{4}$, find a.

36. Let $u = \int_0^{\pi/2} \cos\left(\frac{2\pi}{3}\sin^2 x\right)dx$ and $v = \int_0^{\pi/2} \cos\left(\frac{\pi}{3}\sin x\right)dx$, then the v/u is

MATRIX MATCH TYPE [3, –1]

37. **Match the following**

Column - I		Column - II	
(A)	The locus of mid-points of focal radius of $y^2 = 4ax$ is	(P)	straight line
(B)	The locus of intersection of tangents to $y^2 = 4ax$ with slopes m_1 & m_2 so that $m_1 - m_2 = c\, m_1 m_2$ (where c is a constant) is	(Q)	circle
(C)	The locus of intersection of tangents to $y^2 = 4a(x + a)$ and $y^2 = 4b(x + b)$ which are perpendicular is	(R)	equal parabola
(D)	The locus of the vertex of a parabola having a given point an focus and touching a given line is	(S)	coaxial & equal parabola.

38.

Column - I		Column - II	
(A)	If there are three non concurrent and non parallel lines, then the maximum number of points which are equidistant from all the three lines are	(P)	1
(B)	In a Δ ABC, co–ordintates of orthocentre, centroid and vertex A are (3, 2), (3, 1) and (1, 2) respectively. Then x-corditnate of vetex B is	(Q)	4
(C)	Number of solutions of the equation $[\sin^{-1} x] = x\,[\,x\,]$, where [.] denotes greatest integer function, is	(R)	–1
(D)	If value of the integral $\int_{-1}^{1} \frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right) dx$ is equal to $\frac{p\pi}{2}$ then value of 'p' is	(S)	3

39.

Column-I		Column-II	
(A)	If {(1, 1), (4, 2) and R (x, 0) be three point such that PR + RQ is minimum, then x is equal to	(P)	6
(B)	The area bounded by the curves max {|x|, |y|} = 1 is equal to	(Q)	2
(C)	The number of circles that touch all the three lines $2x - y = 5$, $x + y = 3$ and $4x - 2y = 7$ is equal to	(R)	3
(D)	If a line segment between the lines $3x + 2y - 15 = 0$ and $x + 2y - 4 = 0$ is bisected by the point (3, 1), then negative reciprocal of the slope of line containing the segment is	(S)	4

40.

Column - I		Column - II	
(A)	If the point(6, k) is closest to the curve $x^2 = 2y$ at (2, 2), then k =	(P)	0
(B)	If the curve $y = px^2 + qx + r$ passes through the point (1, 2) and touches the line y = x at the origin, then the value of p – q + r =	(Q)	–2
(C)	Let $f(x) = kx^3 + 9kx^2 + 9x + 3$ be a strictly increasing function and has no stationary point. The greatest value of k is	(R)	1
(D)	Let $0 < a < b < \frac{\pi}{2}$. If $f(x) = \begin{vmatrix} \sin x & \sin a & \sin b \\ \cos x & \cos a & \cos b \\ \tan x & \tan a & \tan b \end{vmatrix}$, then the minimum possible number of roots of f'(x) = 0, lying in (a, b), is	(S)	does not exist

Answer Sheet

Student Name: ______________________________ **Batch : R** **Date : 25/01/15**

1. Ⓐ Ⓑ Ⓒ Ⓓ	2. Ⓐ Ⓑ Ⓒ Ⓓ	3. Ⓐ Ⓑ Ⓒ Ⓓ	4. Ⓐ Ⓑ Ⓒ Ⓓ
5. Ⓐ Ⓑ Ⓒ Ⓓ	6. Ⓐ Ⓑ Ⓒ Ⓓ	7. Ⓐ Ⓑ Ⓒ Ⓓ	8. Ⓐ Ⓑ Ⓒ Ⓓ
9. Ⓐ Ⓑ Ⓒ Ⓓ	10. Ⓐ Ⓑ Ⓒ Ⓓ	11. Ⓐ Ⓑ Ⓒ Ⓓ	12. Ⓐ Ⓑ Ⓒ Ⓓ
13. Ⓐ Ⓑ Ⓒ Ⓓ	14. Ⓐ Ⓑ Ⓒ Ⓓ	15. Ⓐ Ⓑ Ⓒ Ⓓ	16. Ⓐ Ⓑ Ⓒ Ⓓ
17. Ⓐ Ⓑ Ⓒ Ⓓ	18. Ⓐ Ⓑ Ⓒ Ⓓ	19. Ⓐ Ⓑ Ⓒ Ⓓ	20. Ⓐ Ⓑ Ⓒ Ⓓ
21. Ⓐ Ⓑ Ⓒ Ⓓ	22. Ⓐ Ⓑ Ⓒ Ⓓ	23. Ⓐ Ⓑ Ⓒ Ⓓ	24. Ⓐ Ⓑ Ⓒ Ⓓ
25. Ⓐ Ⓑ Ⓒ Ⓓ	26. Ⓐ Ⓑ Ⓒ Ⓓ		

27. ⓪ ① ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨	28. ⓪ ① ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨
29. ⓪ ① ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨	30. ⓪ ① ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨
31. ⓪ ① ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨	32. ⓪ ① ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨
33. ⓪ ① ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨	34. ⓪ ① ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨
35. ⓪ ① ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨	36. ⓪ ① ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

37. A	B	C	D	38. A	B	C	D	39. A	B	C	D	40. A	B	C	D
Ⓟ	Ⓟ	Ⓟ	Ⓟ	Ⓟ	Ⓟ	Ⓟ	Ⓟ	Ⓟ	Ⓟ	Ⓟ	Ⓟ	Ⓟ	Ⓟ	Ⓟ	Ⓟ
Ⓠ	Ⓠ	Ⓠ	Ⓠ	Ⓠ	Ⓠ	Ⓠ	Ⓠ	Ⓠ	Ⓠ	Ⓠ	Ⓠ	Ⓠ	Ⓠ	Ⓠ	Ⓠ
Ⓡ	Ⓡ	Ⓡ	Ⓡ	Ⓡ	Ⓡ	Ⓡ	Ⓡ	Ⓡ	Ⓡ	Ⓡ	Ⓡ	Ⓡ	Ⓡ	Ⓡ	Ⓡ
Ⓢ	Ⓢ	Ⓢ	Ⓢ	Ⓢ	Ⓢ	Ⓢ	Ⓢ	Ⓢ	Ⓢ	Ⓢ	Ⓢ	Ⓢ	Ⓢ	Ⓢ	Ⓢ

ANSWER WITH SOLUTION

ANSWER KEY

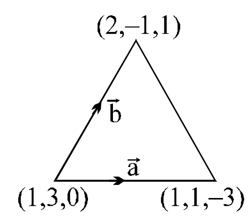
Q.	1	2	3	4	5	6	7	8	9	10	11	12
A.	B	C	B	C	A	B	D	B	C	B	C	A
Q.	13	14	15	16	17	18	19	20	21	22	23	24
A.	C	B	A	B	AC	All	ABD	BC	AC	ABC	AD	All
Q.	25	26	27	28	29	30	31	32	33	34	35	36
A.	AB	BC	2	2	3	2	8	4	1	3	2	2

Q.	37.	38.
A.	(A) – (S), B – (S), (C) – (P), (D) – (Q)	(A) – (Q), B – (Q), (C) – (P), (D) – (R)
Q.	39.	40.
A.	(A) – (Q), B – (S), (C) – (Q), (D) – (P)	(A) – (P), B – (P), (C) – (S), (D) – (R)

SOLUTION

3. $P_0 = \left(\frac{4}{3}, 1, \frac{-2}{3}\right)$

vector normal to the plane



$$\vec{n} = \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -2 & -3 \\ 1 & -4 & 1 \end{vmatrix} = \hat{i}(-2-12) - \hat{j}(0+3) + \hat{k}(0+2)$$

$$= -(14\hat{i} + 3\hat{j} - 2\hat{k})$$

equation of the line is $\vec{r} = \frac{4}{3}\hat{i} + \hat{j} - \frac{2}{3}\hat{k} + \lambda(14\hat{i} + 3\hat{j} - 2\hat{k})$ Ans.]

5. $(x - x_1)^2 + (y - y_1)^2 = r^2$

x-axis intercept = 2a $\Rightarrow$ $2\sqrt{x_1^2 - (x_1^2 + y_1^2 - r^2}$ = 2a

$r^2 - y_1^2 = a^2$

$\therefore$ $y_1^2 = r^2 - a^2$

Passes through (0, b)

$\therefore$ $x_1^2 + (b - y_1)^2 = r^2 \Rightarrow x_1^2 + b^2 - 2by_1 + y_1^2 - r^2 = 0 \Rightarrow x_1^2 + b^2 - 2by_1 - a^2 = 0$

$\therefore$ Locus of (x_1, y_1) will be $x^2 - 2by + b^2 = a^2$]

6. B = $A^{2^n} = A^{2\cdot 2^{n-1}} = (A^2)^{2^{n-1}} = (A^{-1})^{2^{n-1}}$

$= \left(A^{2^{n-1}}\right)^{-1} = \left(A^{2\cdot 2^{n-2}}\right)^{-1} = \left((A^2)^{2^{n-2}}\right)^{-1} = \left((A^{-1})^{-1}\right)^{2^{n-2}} = A^{2^{(n-2)}}$

so B = C $\Rightarrow$ A, B, C are answers]

7. $(\hat{i} + \hat{j}) \times (\hat{j} + \hat{k}) = \hat{i} - \hat{j} + \hat{k}$ $\Rightarrow$ unit vector perpendicular as to the plane of

$\hat{i} + \hat{j}$ and $\hat{j} + \hat{k}$ is $\frac{1}{\sqrt{3}}(\hat{i} - \hat{j} + \hat{k})$ similarly other two unit vectors are

$\frac{1}{\sqrt{3}}\left(\hat{i}+\hat{j}-\hat{k}\right)$ and $\frac{1}{\sqrt{3}}\left(-\hat{i}+\hat{j}+\hat{k}\right) \Rightarrow$ V = $[\hat{n}_1\ \hat{n}_2\ \hat{n}_3] = \frac{1}{3\sqrt{3}}\begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{vmatrix} = \frac{4}{3\sqrt{3}}$

Alternatively : Let $\vec{a} = \hat{i}+\hat{j}$; $\vec{b} = \hat{j}+\hat{k}$ & $\vec{c} = \hat{k}+\hat{i}$.

Now $\left[\vec{a}\times\vec{b}\ ,\ \vec{b}\times\vec{c}\ ,\ \vec{c}\times\vec{a}\right] = \left[\vec{a}\ \vec{b}\ \vec{c}\right]^2 = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}^2 = \left[1(1)-1(0-1)\right]^2 = 4$

Hence actual volume with unit vectors = $\frac{4}{\left|\vec{a}\times\vec{b}\right|\left|\vec{b}\times\vec{c}\right|\left|\vec{c}\times\vec{a}\right|}$

Now $\left|a\times\vec{b}\right| = \sqrt{\vec{a}^2\ \vec{b}^2-(\vec{a}.\vec{b})^2} = \sqrt{4-1} = \sqrt{3}$ etc $V_{actual} = \frac{4}{3\sqrt{3}}$]

8. L = $\lim_{n\to\infty} n\int_0^{\pi/2}\left(1-\sqrt[n]{\sin x}\right)dx$; put $n = \frac{1}{t}$

$= \lim_{t\to 0^+}\int_0^{\pi/2}\left(\frac{1-(\sin x)^t}{t}\right)dx$; L = $\lim_{t\to 0}\int_0^{\pi/2}\left(\frac{1-e^{t\,ln(\sin x)}}{t\,ln(\sin x)}\cdot ln(\sin x)\right)dx$

but $\lim_{t\to 0}\frac{1-e^{t\,ln(\sin x)}}{t\,ln(\sin x)} = -1$; $\therefore$ L = $-\int_0^{\pi/2} ln(\sin x)\,dx = \frac{\pi}{2}\,ln\,2$ Ans.]

11-12. Point P is (1, 2)

Tangent is 2y = 2(x + 1)

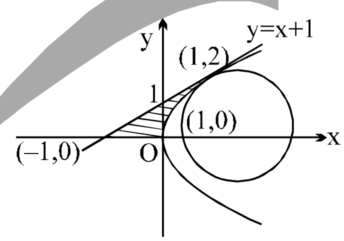
i.e. y = x + 1 ....(1)

hence image of $y^2 = 4x$ in (2) can be written as

$(x+1)^2 = 4(y-1)$ $\Rightarrow$ **(C)**

Find the image of $(t^2, 2t)$ in the tangent line and then eliminate *t* to get the image

Area = $\int_0^2\left[\frac{y^2}{4}-(y-1)\right]dy = \left.\frac{y^3}{12}-\frac{y^2}{2}+y\right]_0^2$



$= \frac{8}{12}-2+2 = \frac{2}{3}$ $\Rightarrow$ **(A)**]

13. $y = e^{a+bx^2}$, passes through (1, 1)

$1 = e^{a+b} \Rightarrow a + b = 0$

also $\left.\frac{dy}{dx}\right|_{(1,1)} = -2$

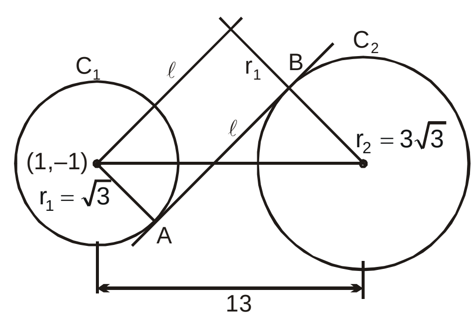
$e^{a+bx^2}\cdot 2bx = -2$

$\Rightarrow$ $e^{a+b}\cdot 2b(1) = -2$

$\Rightarrow$ b = –1 and a = 1

$\Rightarrow$ (a, b) = (1, –1)

hence $C_1 : (x-6)^2+(y+1)^2 = (\sqrt{3})^2$, $C_2 : (x-6)^2+(y-11)^2 = (3\sqrt{3})^2$] C_1 & C_2 are separated



$AB^2 = \ell^2 = d^2-(r_1+r_2)^2 = 169-(4\sqrt{3})^2 = 121$

AB = 11 Ans.

14. again $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$; f(1) = 1

$$= \frac{f(x)\left[\frac{f(x+h)}{f(x)}-1\right]}{h} = \frac{f(x)}{x}\lim_{h\to 0}\left[\frac{f\left(1+\frac{h}{x}\right)-1}{\frac{h}{x}}\right] = \frac{f'(1).f(x)}{x} = \frac{2f(x)}{x} \text{ (as } f(1) = f^2(1) \text{ but } f(1) \neq 0$$

$\Rightarrow$ f(1) = 1)

$$\frac{f'(x)}{f(x)} = \frac{2}{x}$$

$\ell n(f(x)) = 2\ \ell n\ x + C$

$x = 1,\ f(1) = 1 \Rightarrow C = 0$

$f(x) = x^2$

$$\therefore \quad I = \int_b^a f(x)d(\ell n\ x) = \int_{-1}^{1} x^2 d(\ell n\ x) = \int_{1/e}^{e} x\,dx = \left.\frac{x^2}{2}\right]_{1/e}^{e} = \frac{e^2 - e^{-2}}{2}$$ **Ans.**

17. $I = \int_0^{\pi/2} f(\sin\theta)\cos\theta\,d\theta$

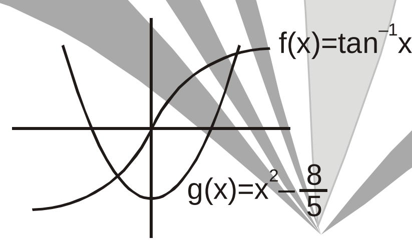
$I = \int_0^{\pi/2} f(\cos\theta)\sin\theta\,d\theta$

$2I = \int_0^{\pi/2} (f(\sin\theta)\cos\theta + f(\cos\theta)\sin\theta)\,d\theta \leq \int_0^{\pi/2} 1\,d\theta = \frac{\pi}{2}$

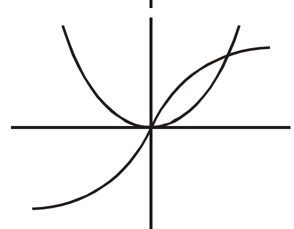
18. (A), (B) (C) and (D) are correct

(B) $f(x) = \begin{bmatrix} 1 & \text{if} & x\in Q \\ -1 & \text{if} & x\notin Q \end{bmatrix}$ then $|f(x)| = 1\ \forall\ x$]

19. (A)



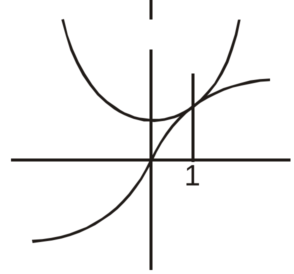
(B)



$f'(0) = 1$

$g'(0) = 0$

(C)



$f'(x) = \frac{1}{2}$.

$g(x) = \frac{1}{4}x^2 + \left(\frac{\pi-1}{4}\right)$

$g(1) = \frac{\pi}{4} = f(1)$

$g'(1) = \frac{1}{2}$

f & g touch each other.

(D) If we pull down the graph of g(x) in (C) slightly down, there will be two solutions.

20. $f(x) = x^3 - x^2 + 100x + 2002$

$f'(x) = 3x^2 - 2x + 100 > 0 \ \ \forall \ x \in R$

$\therefore$ f (x) is increasing (strictly)

$\therefore \quad f\left(\frac{1}{2000}\right) > f\left(\frac{1}{2001}\right)$

Also $f(x-1) > f(x-2)$ as $x - 1 > x - 2$ for $\forall$ x]

21. $\vec{a} \times (\vec{b} \times \vec{c}) + (\vec{a} \cdot \vec{b})\vec{b} = (4 - 2x - \sin y)\ \vec{b} + (x^2 - 1)\vec{c}$

$\Rightarrow \quad (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} + (\vec{a} \cdot \vec{b})\vec{b} = (4 - 2x - \sin y)\ \vec{b} + (x^2 - 1)\vec{c}$

Now $(\vec{c} \cdot \vec{c})\vec{a} = \vec{c}$

$\Rightarrow \quad (\vec{a} \cdot \vec{c})\ (\vec{c} \cdot \vec{c}) = (\vec{c} \cdot \vec{c})$

$\Rightarrow \quad \vec{a} \cdot \vec{c} = 1$

$\Rightarrow \quad 1 + \vec{a} \cdot \vec{b} = 4 - 2x - \sin y,\ x^2 - 1 = -(\vec{a} \cdot \vec{b})$

$\Rightarrow \quad 1 = 4 - 2x - \sin y + x^2 - 1$

$\Rightarrow \quad \sin y = x^2 - 2x + 2 = (x - 1)^2 + 1$

But $\sin y \le 1$, therefore $x = 1$, $\sin y = 1$

$\Rightarrow \quad y = (4n + 1)\frac{\pi}{2},\ n \in I$

22. Let $Z = a + ib$, $b \neq 0$ where Im Z = b

$Z^5 = (a + ib)^5 = a^5 + {}^5C_1\, a^4 bi + {}^5C_2\, a^3\, b^2 i^2 + {}^5C_3\, a^2 b^3\, i^3 + {}^5C_4 ab^4 i^4 + i^5 b^5$

$\text{Im } Z^5 = 5a^4 b - 10a^2 b^3 + b^5$

$y = \frac{\text{Im}\, Z^5}{\text{Im}^5\, Z} = 5\left(\frac{a}{b}\right)^4 - 10\left(\frac{a}{b}\right)^2 + 1$

Let $\left(\frac{a}{b}\right)^2 = x$ (say), $x \in R^+$

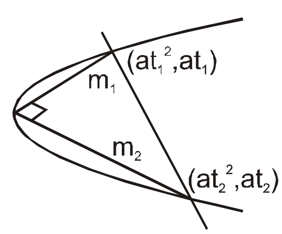
$y = 5x^2 - 10x + 1 = 5[x^2 - 2x] + 1 = 5[(x - 1)^2] - 4$

Hence $y_{min} = -4$ **Ans.**]

25. $\frac{dy}{dx} = \frac{(y - x) \pm \sqrt{(x - y)^2 + 4xy}}{2y}$

$\Rightarrow \quad \frac{dy}{dx} = 1 \quad \text{or} \quad \frac{dy}{dx} = -\frac{x}{y}$]

27. 2



$y + t_1 x = 2\,a\,t_1 + a\,t_1^3$

$\Rightarrow \quad 2\,a\,t_2 + a\,t_1 t_2^2 = 2\,a\,t_1 + a\,t_1^3$

$\Rightarrow \quad 2\,t_2 + t_1 t_2^2 = 2\,t1 + t_1^3$

also $m_1 \times m_2 = -1$

$\frac{2}{t_1} + \frac{2}{t_2} = -1 \quad \Rightarrow \quad t_2 = -\frac{4}{t_1}$

$\Rightarrow \quad -\frac{8}{t_1} + \frac{16}{t_1} = 2\,t + t_1^3$

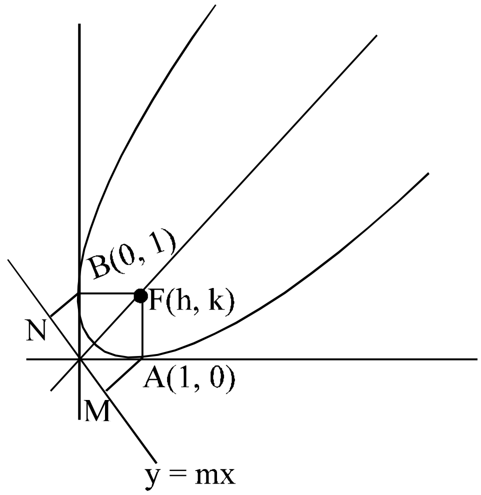
$\Rightarrow \quad t_1^4 + 2\,t_1^2 - 8 = 0$

$\Rightarrow \quad (t_1^2 + 4)\,(t_1^2 - 2) = 0$

$\Rightarrow \quad t_1^2 = 2$

28. 2

Since the x-axis and y-axis are two perpendicular tangents to the parabola and both meet at the origin, the directrix passes through the origin.



Let y = mx be the directrix and (h, k) be the focus. FA = AM

$\Rightarrow \quad \sqrt{(h-1)^2 + k^2} = \left|\frac{m}{\sqrt{1+m^2}}\right|$ ....(1)

and FB = BN

$\Rightarrow \quad \sqrt{h^2 + (k-1)^2} = \left|\frac{m}{\sqrt{1+m^2}}\right|$ ....(2)

From equations (1) and (2), we get

$(h - 1)^2 + h^2 + k^2 + (k - 1)^2 = 1$

$\Rightarrow \quad 2x^2 - 2x + 2y^2 - 2y + 1 = 0$ is the required locus.]

29. Ans. 3

$f(x) = \begin{cases} a\cos x + b\sin x & \text{if } 0 \le x \le \frac{\pi}{2} \\ a\cos x - b\sin x & \text{if } -\frac{\pi}{2} \le x \le 0 \end{cases}$ [**Ans.** $a = -\frac{1}{\sqrt{\pi+\sqrt{3}}}$; $b = -\frac{\sqrt{3}}{\sqrt{\pi+\sqrt{3}}}$]

for $-\pi/2 < x < 0$

$f'(x) = -a\sin x - b\cos x$ ....(1)

and $f''(x) = -a\cos x + b\sin x$ ....(2)

since f (x) has a minima at $x = -\pi/3$

hence $f'(-\pi/3) = 0$ and $f''(-\pi/3) > 0$

now $f'(-\pi/3) = +a \cdot \frac{\sqrt{3}}{2} - \frac{b}{2} = 0 = \sqrt{3}a - b = 0$

and $f''(-\pi/3) = -\frac{a}{2} - b \cdot \frac{\sqrt{3}}{2} = -\frac{1}{2}[a + b\sqrt{3}] = -2a > 0$

hence a < 0 and b < 0

now $I = \int_{-\pi/2}^{\pi/2} (f(x))^2\,dx = \int_{-\pi/2}^{0} f^2(x)\,dx + \int_{0}^{\pi/2} f^2(x)\,dx$

$= \int_{-\pi/2}^{0} (a^2\cos^2 x - 2ab\sin x\cos x + b^2\sin^2 x)\,dx + \int_{0}^{\pi/2} (a^2\cos^2 x + 2ab\sin x\cos x + b^2\sin^2 x)\,dx$

hence $I = \frac{\pi a^2}{2} + \frac{\pi b^2}{2} + 2ab = 2$

$2(\sqrt{3}+\pi)a^2 = 2 \quad \Rightarrow \quad \boxed{a = -\frac{1}{\sqrt{\pi+\sqrt{3}}}}$ and $\boxed{b = -\frac{\sqrt{3}}{\sqrt{\pi+\sqrt{3}}}}$ Ans.]

31. 8

Put $4\sqrt{\cos\frac{x}{2}} - 5 - \frac{\sqrt{2}}{2} = y$ and form a quadratic in y . Solve it for y .

Ans. : $x = 4n\pi$, $n \in I$

32. Ans 4

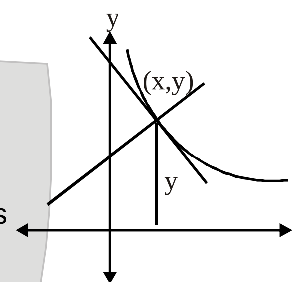
$$\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ [x] & [y] & [z]+1 \end{vmatrix}$$

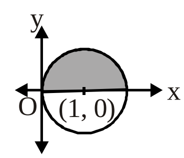
solving = [x] + [y] + [z] + 1

taking maximum value we get 4

note that [x] is always an integer.] [Ans. 1]

33. Equation of normal is





$Y - y = -\frac{1}{m}(X - x)$

$X + mY - (x + my) = 0$ ...... (1)

Perpendicular distance from (0, 0) to equation (1) is

$\left|\frac{x+my}{\sqrt{1+m^2}}\right| = |y|$

$\Rightarrow (x + my)^2 = y^2(1 + m^2) \Rightarrow x^2 + 2mxy = y^2 \Rightarrow m = \frac{y^2 - x^2}{2xy} \Rightarrow 2xy\frac{dy}{dx} = y^2 - x^2$ ....(2)

Put $y^2 = t \Rightarrow 2y\frac{dy}{dx} = \frac{dt}{dx}$

$\therefore$ Equation (2) becomes

$$x\frac{dt}{dx} = t - x^2 \Rightarrow \frac{dt}{dx} - \frac{1}{x}t = -x$$

$\therefore$ I.F. $= e^{-\int\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$

Now general solution is given by

$$t\left(\frac{1}{x}\right) = -x + C \Rightarrow y^2\left(\frac{1}{x}\right) = -x + C$$

As (1, 1) satisfy it, so C = 2

$\Rightarrow y^2 = -x^2 + 2x \Rightarrow x^2 + y^2 - 2x = 0$

Hence required area $= \frac{k\pi}{2}$

$\therefore$ k = 1 **Ans.**]