MATHEMATICS

Target IIT-JEE 2016

Class XI

STRAIGHT LINE

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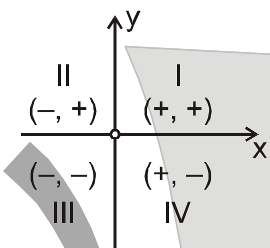
STRAIGHT LINE

A. RECTANGULAR (CARTESIAN) COORDINATES IN A PLANE

Let us draw in the plane two mutually perpendicular intersecting lines Ox and Oy which are termed coordinate axes. The point of intersection O of the two axes is called the origin of coordinates, or simply the origin. It divides each of the axes into two semi-axes. One of the semi-axes is conventionally called positive (indicated by an arrow in the drawing), the other being negative.

Any point A in a plane is specified by a pair of numbers called the rectangular coordinates of the point A – the abscissa (x) and the ordinate (y) according to the following rule.

Through the point A we draw a straight line parallel to the axis of ordinates (Oy) to intersect the axis of abscissas (Ox) at some point A_x. The abscissa of the point A should be understood as a number x whose absolute value is equal to the distance from O to A_x which is positive if A_x belongs to the positive semi-axis and negative if A_x belongs to the negative semi-axis. If the point A_x coincides with the origin, then we put x equal to zero.



The ordinate (y) of the point A is determined in a similar way.

We shall use the following notation : A(x, y) which means that the coordinates of the point A are x (abscissa) and y (ordinate).

Ex. 1 ABCD is a square, having it's vertices A and B on the positive x and y axis respectively. Given that $C \equiv (12, 17)$, find the coordinates of all the vertices.

Sol. Let the side length of the square be 'a' and $\angle BAQ = \theta$

$\Rightarrow \quad \angle C_1BC = \angle D_1DA = \theta$

$\Rightarrow \quad A \equiv (a \cos \theta, 0),\ B \equiv (0, a \sin \theta)$

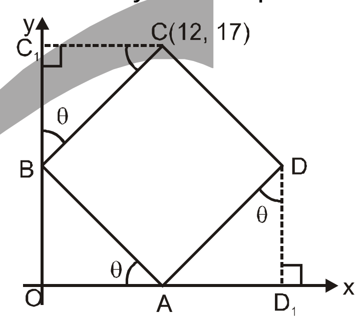
$C \equiv (a \sin\theta, a \sin \theta + a \cos \theta)$ and

$D \equiv (a \cos\theta + a \sin \theta, a \cos \theta)$

Thus, $a \sin\theta = 12$, $a \sin \theta + a \cos \theta = 17$

$\Rightarrow \quad a \cos \theta = 5$

$\Rightarrow \quad A \equiv (5, 0),\ B \equiv (0, 12),\ C \equiv (12, 17),\ D \equiv (17, 5).$



B. POLAR COORDINATES

In this system of coordinates the position of a point is determined by its distance from a fixed point O, usually called the pole (though it might equally well be called the origin), and the angle which the line joining the pole to the point makes with a fixed line through the pole, called the initial line. Thus if OA be the initial line, the polar coordinates of a point P are OP which is known as the radius vector, and the angle AOP which is called the vectorial angle. The vectorial angle is measured from the initial lines as in Trigonometry ; it is usually considered positive if measured round from OA in the opposite direction to that of the rotation of the hands of a watch, and negative in the other direction. But it may on occasion be more convenient to take to rotation positive in the same direction as that of the hands of a watch. To mark a point whose polar coordinates (r, θ) are given, we first measure the vectorial angle θ and then cut off the radius vector (=r). The extremity P of this is the point (r, θ).

Formulae connecting the Polar and Cartesian coordinates of a point.

It is to be understood in what follows that the pole and the initial line in the polar system are respectively the origin and the axis of x in the Cartesian system, and the positive direction of measurement of the vectorial angle is towards the axis of y.

Let (x, y) be the Cartesian co-ordinates of a point P, (r, θ) its polar coordinates.

First, let the Cartesian axes be rectangular.

We have $x = r\cos\theta$, $y = r\sin\theta$,

and these formulae hold in whichever quadrant P may be.

From the above we have $r^2 = x^2 + y^2$, $\tan\theta = \dfrac{y}{x}$,

Ex. 2 Change to polar co-ordinates the equation $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

Sol. The given equation is $(x^2 + y^2)^2 = a^2(x^2 - y^2)$ or $(r^2)^2 = a^2(r^2\cos^2\theta - r^2\sin^2\theta)$

or $r^4 = a^2r^2(\cos^2\theta - \sin^2\theta)$ or $r^2 = a^2\cos 2\theta$.

Ex. 3 Transform to Cartesian co-ordinates the equation $r(\cos 3\theta + \sin 3\theta) = 5k\sin\theta\cos\theta$.

Sol. The given equation is $r(\cos 3\theta + \sin 3\theta) = 5k\sin\theta\cos\theta$

or $r(4\cos^3\theta - 3\cos\theta + 3\sin\theta - 4\sin^3\theta)\ 5k\sin\theta\cos\theta$

or $4r(\cos^3\theta - \sin^3\theta) - 3r(\cos\theta - \sin\theta) = 5k\sin\theta\cos\theta$

Multiplying both sides by r^2, we get

$4(r^3\cos^3\theta - r^3\sin^3\theta) - 3r^2(r\cos\theta - r\sin\theta) = 5k\, r\sin\theta.\, r\cos\theta$

or $4(x^3 - y^3) - 3(x^2 + y^2)(x - y) = 5k.yx$

or $4x^3 - 4y^3 - 3x^3 + 3x^2y - 3y^2x + 3y^3 = 5kxy$

or $x^3 + 3x^2y - 3xy^2 - y^3 = 5kxy$. Ans.

C. DISTANCE FORMULA

The distance between the points $A(x_1,y_1)$ and $B(x_2,y_2)$ is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Section Formula :

If P(x , y) divides the line joining $A(x_1, y_1)$ & $B(x_2, y_2)$ in the ratio m : n, then ;

$x = \dfrac{mx_2 + nx_1}{m + n}$; $y = \dfrac{my_2 + ny_1}{m + n}$.

If $\dfrac{m}{n}$ is positive, the division is internal, but if $\dfrac{m}{n}$ is negative, the division is external.

Note : If P divides AB internally in the ratio m : n & Q divides AB externally in the ratio m : n then P & Q are said to be harmonic conjugate of each other w.r.t. AB.

Mathematically ; $\dfrac{2}{AB} = \dfrac{1}{AP} + \dfrac{1}{AQ}$ i.e. AP, AB & AQ are in H.P.

Centroid and Incentre :

If $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are the vertices of triangle ABC, whose sides BC, CA, AB are of lengths a, b, c respectively, then the coordinates of the centroid are :

$\left(\dfrac{x_1 + x_2 + x_3}{3}, \dfrac{y_1 + y_2 + y_3}{3}\right)$ & the coordinates of the incentre are ;

$\left(\dfrac{ax_1 + bx_2 + cx_3}{a + b + c}, \dfrac{ay_1 + by_2 + cy_3}{a + b + c}\right)$

Note that incentre divides the angle bisectors in the ratio

(b + c) : a ; (c + a) : b & (a + b) : c.

Note :

(i) Orthocentre , Centroid & circumcentre are always collinear & centroid divides the line joining orthocentre & cercumcentre in the ratio 2 : 1.

(ii) In an isosceles triangle incentre,orthocentre , centroid & circumcentre lie on the same line.

Ex. 4 The line joining the points (1, –2) and (–3, 4) is trisected: find the co-ordinates of the points of trisection.

Sol. Let the points (1, –2) and (–3, 4) be A and B respectively. If P and Q be the points of trisection of AB, the ratio AP : PB will be equal to 1 : 2 and the ratio AQ : QB will be equal to 2 : 1. Hence the co-ordinates of P are

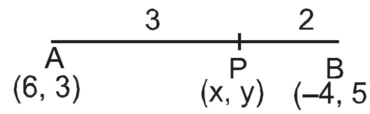
$$\left[\frac{(1\times-3)+(2\times1)}{1+2},\frac{(1\times4)+(2\times-2)}{1+2}\right] \text{ or } \left(-\frac{1}{3},0\right)$$

Co-ordinates of Q are

$$\left[\frac{(2\times-3)(1\times1)}{2+1},\frac{(2\times4)+(1\times-2)}{2+1}\right] \text{ or } \left(-\frac{5}{3},2\right)$$

Ex. 5 Find the coordinates of the point which divides the line segment joining the points (6, 3) and (–4, 5) in the ratio 3 : 2 internally and (ii) externally.

Sol. Let P(x, y) be the required point.

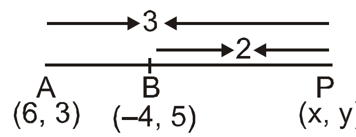


(i) For internal division :

$$x = \frac{3x-4+2\times6}{3+2} \text{ and } y = \frac{3\times5+2\times3}{3+2} \text{ or } x = 0 \text{ and } y = \frac{21}{5}$$

So the coordinates of P are $\left(0,\frac{21}{5}\right)$ Ans.

(ii) For external division



$$x = \frac{3\times-4-2\times6}{3-2} \text{ and } y = \frac{3\times5-2\times3}{3-2}$$

or $x = -24$ and $y = 9$

So the coordinates of P are (–24, 9) Ans.

Ex. 6 Find the coordinates of (i) centroid (ii) in-centre of the triangle whose vertices are (0, 6), (8, 12) and (8, 0).

Sol.

(i) We know that the coordinates of the centroid of a triangle whose angular points are (x_1, y_1), (x_2, y_2) (x_3, y_3) are

$$\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right)$$

So the coordinates of the centroid of a triangle whose vertices are (0, 6), (8, 12) and (8, 0) are

$$\left(\frac{0+8+8}{3},\frac{6+12+0}{3}\right) \text{ or } \left(\frac{16}{3},6\right) \quad \text{Ans.}$$

(ii) Let A(0, 6), B(8, 12) and C(8, 0) be the vertices of triangle ABC.

Then $c = AB = \sqrt{(0-8)^2+(6-12)^2} = 10$, $b = CA = \sqrt{(0-8)^2+(6-0)^2} = 10$

and $a = BC = \sqrt{(8-8)^2+(12-0)^2} = 12$

The co-ordinates of the in-centre are $\left(\frac{ax_1+bx_2+cx_3}{a+b+c},\frac{ay_1+by_2+cy_3}{a+b+c}\right)$

or $$\left(\frac{12\times0+10\times8+10\times8}{12+10+10},\frac{12\times6+10\times12+10\times0}{12+10+10}\right)$$

or $$\left(\frac{160}{32},\frac{192}{32}\right) \text{ or } (5, 6) \quad \text{Ans.}$$

Ex. 7 The co–ordinates of the vertices of a triangle are (x_1, y_1), (x_2, y_2) and (x_3, y_3). The line joining the first two is divided in the ratio l : k, and the line joining this point of division to the opposite angular point is then divided in the ratio m : k + l. Find the co-ordinates of the latter point of section.

Sol. The co-ordinates of the vertices are given to be (x_1, y_1), (x_2, y_2) and (x_3, y_3). Let these points be A, B and C respectively. The co-ordinates of the point P which divides AB in the ratio l : k will be

$$\left(\frac{lx_2 + kx_1}{l + k}, \frac{ly_2 + ky_1}{l + k}\right)$$

The co-ordinates of the point Q which divides the join of P and C in the ratio m : (k +l) will be

$$\left\{\frac{mx_3 + (k + l)\dfrac{lx_2 + kx_1}{(l + k)}}{m + l + k}, \frac{my_3 + (k + l)\dfrac{ly_2 + ky_1}{(l + k)}}{m + l + k}\right\} \quad \text{or} \quad \left(\frac{kx_1 + lx_2 + mx_3}{k + l + m}, \frac{ky_1 + ly_2 + my_3}{k + l + m}\right)$$

Ex. 8 The quadratic equations , $a x^2 + b x + c = 0$ & $A x^2 + B x + C = 0$ have roots x_1 , x_2 and x_3 , x_4. If the points $(x_1 , 0)$ and $(x_2 , 0)$ divide the line joining $(x_3 , 0)$ & $(x_4 , 0)$ internally and externally in the same ratio then show that , $2 (c A + C a) = b B$.

Sol. $x_1 + x_2 = -\frac{b}{a}$; $x_1 x_2 = \frac{c}{a}$ $\qquad$ $x_3 + x_4 = -\frac{B}{A}$; $x_3 x_4 = \frac{C}{A}$

$x_1 = \dfrac{\lambda x_4 + x_3}{\lambda + 1} \Rightarrow \lambda x_4 + x_3 = (\lambda + 1) x_3$; $\qquad$ $x_1 = \dfrac{\lambda x_4 - x_3}{\lambda - 1} \Rightarrow \lambda x_4 - x_3 = (\lambda - 1) x_2$

$\therefore \lambda (x_4 - x_1) = x_1 - x_3$; $\qquad$ $\lambda (x_4 - x_2) = x_3 - x_2$

dividing cross multiplying and rearranging , $2 (x_1 x_2 + x_3 x_4) = (x_3 + x_4) (x_1 + x_2)$]

Slope of a Line :

If θ is the angle at which a straight line is inclined to the positive direction of x–axis, & $0° \le \theta < 180°$, $\theta \ne 90°$, then the slope of the line, denoted by m, is defined by $m = \tan\theta$. If θ is 90°, m does not exist, but the line is parallel to the y–axis.

If $\theta = 0$, then m = 0 & the line is parallel to the x–axis.

If A (x_1, y_1) & B (x_2, y_2), $x_1 \ne x_2$, are points on a straight line, then the slope m of the line is given by : $m = \left(\dfrac{y_1 - y_2}{x_1 - x_2}\right)$.

Area of a Triangle :

If (x_i, y_i), i = 1, 2, 3 are the vertices of a triangle, then its area is equal to $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$,

provided the vertices are considered in the counter clockwise sense. The above formula will give a (–) ve area if the vertices (x_i, y_i) , i = 1, 2, 3 are placed in the clockwise sense.

Condition Of Collinearity Of Three Points – (Slope Form) :

Points A (x_1, y_1), B (x_2, y_2), C(x_3, y_3) are collinear if $\left(\dfrac{y_1 - y_2}{x_1 - x_2}\right) = \left(\dfrac{y_2 - y_3}{x_2 - x_3}\right)$.

Condition Of Collinearity Of Three Points – (Area Form) :

The points (x_i, y_i), i = 1 , 2 , 3 are collinear if $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.

Ex. 9 If the coordinates of two points A and B are (3, 4) and (5, –2) respectively. Fnd the coordinates of any point P if AP = PB and Area of ΔPAB = 10.

Sol. Let the coordinates of P be (x, y). Then

PA = PB $\Rightarrow$ $PA^2 = PB^2$ $\Rightarrow$ $(x-3)^2 + (y-4)^2 = (x-5)^2 + (y+2)^2$

$\Rightarrow$ $x - 3y - 1 = 0$

Now, Area of ΔPAB = 10 $\Rightarrow$ $\dfrac{1}{2}\begin{vmatrix} x & y & 1 \\ 3 & 4 & 1 \\ 5 & -2 & 1 \end{vmatrix} = \pm 10 \Rightarrow 6x + 2y - 26 = \pm 20$

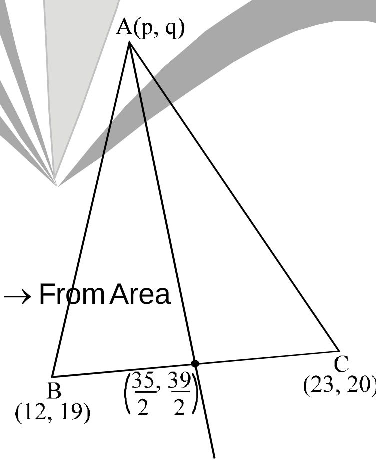
$\Rightarrow$ $6x + 2y - 46 = 0$ or $6x + 2y - 6 = 0$

$\Rightarrow$ $3x + y - 23 = 0$ or $3x + y - 3 = 0$

Solving x – 3 y – 1 = 0 and 3x + y – 23 we get x = 7, y = 2. Solving x – 3y – 1 = 0 and 3x + y – 3 = 0, we get x = 1, y = 0. Thus, the coordinates of P are (7, 2) or (1, 0) Ans.

Ex. 10 Triangle ABC lies in the Cartesian plane and has an area of 70 sq. units. The coordinates of B and C are (12, 19) and (23, 20) respectively and the coordinates of A are (p, q). The line containing the median to the side BC has slope –5. Find the largest possible value of (*p* + *q*).

Sol. $\dfrac{(39/2)-q}{(35/2)-p} = -5$ From slope



$39 - 2q = -5(35 - 2p)$

$39 - 2q = -175 + 10p$

i.e. $\boxed{5p + q = 107}$

Also,

$$\pm\begin{vmatrix} p & q & 1 \\ 12 & 19 & 1 \\ 23 & 20 & 1 \end{vmatrix} = 140 \rightarrow \text{From Area}$$

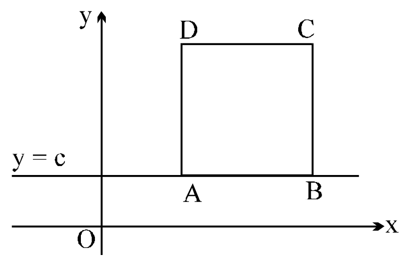
i.e.

	$11q - p = 337$	$11q - p = 57$
Also	$\underbrace{5p + q = 107}_{\text{solving}}$	$\underbrace{5p + q = 107}_{\text{solving}}$
$\therefore$	p = 15 & q = 32	p = 20 & q = 7
So,	p + q = 47	p + q = 27

Hence, largest possible value = 47

Ex. 11 A square ABCD lying in I-quadrant has area 36 sq. units and is such that its side AB is parallel to x-axis. Vertices A, B and C are on the graph of $y = \log_a x$, $y = 2\log_a x$ and $y = 3\log_a x$ respectively then find the value of 'a'.

Sol. AB : y = c (c > 0)



Length of the side of square = 6

A has y-coordinate = c and it lies on $y = \log_a x$

$\therefore$ x-coordinate = a^c

$\therefore$ point A is (a^c, c)

|||ly B is $(a^{c/2}, c)$

and BC $\perp$ AB

$\therefore$ C has x-coordinate = $a^{c/2}$ and it lies on $y = 3\log_a x$

$y = 3\log_a a^{c/2} = \dfrac{3c}{2}$ $\quad\therefore\quad$ point C is $\left(a^{c/2}, \dfrac{3c}{2}\right)$

$|AB| = 6$ $\quad\therefore\quad$ $a^c - a^{c/2} = 6$ $\quad a > 0, c > 0$

let $a^{c/2} = t$ $\quad t^2 - t - 6 = 0 \Rightarrow (t-3)(t+2) = 0 \Rightarrow t = 3$ or $t = -2$ (rejected)

$\therefore\ t = 3 \Rightarrow a^{c/2} = 3 \Rightarrow a^c = 9$

also $|BC| = 6$ $\quad \dfrac{3c}{2} - c = 6;\quad \therefore\quad c = 12$

$\therefore\ a^{12} = 9 \Rightarrow a^6 = 3;\quad \therefore\quad a = \sqrt[6]{3}$

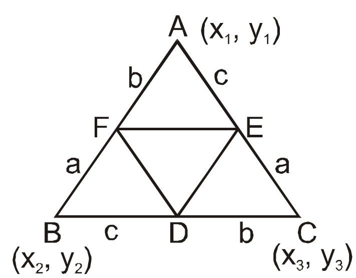
Ex. 12 The internal bisectors of the angles of a triangle ABC meet the sides in D, E and F respectively. Show that the area of the triangle DEF is equal to

$$\frac{2\Delta abc}{(a+b)(b+c)(c+a)}$$, where Δ denotes the area of the triangle ABC.

Sol. D $\left(\dfrac{bx_2 + cx_3}{b+c}, \dfrac{by_2 + cy_3}{b+c}\right)$

E $\left(\dfrac{ax_1 + cx_3}{a+c}, \dfrac{ay_1 + cy_3}{a+c}\right)$

F $\left(\dfrac{ax_1 + bx_2}{a+b}, \dfrac{ay_1 + by_2}{a+b}\right)$



$$\Delta DEF = \frac{1}{2(a+b)(b+c)(c+a)}\begin{vmatrix} ax_1 + cx_3 & ay_1 + cy_3 & a+c \\ bx_2 + ax_1 & by_2 + ay_1 & b+a \\ bx_2 + cx_3 & by_2 + cy_3 & b+c \end{vmatrix}$$

$$= \frac{1}{2(a+b)(b+c)(c+a)}\begin{vmatrix} a & c & 0 \\ a & 0 & b \\ 0 & c & b \end{vmatrix}\begin{vmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} \Rightarrow \text{Result}$$

Ex. 13 If the point $\left(\dfrac{a^3}{a-1}, \dfrac{a^2-3}{a-1}\right)$; $\left(\dfrac{b^3}{b-1}, \dfrac{b^2-3}{b-1}\right)$ & $\left(\dfrac{c^3}{c-1}, \dfrac{c^2-3}{c-1}\right)$ are collinear for three distinct values of a, b & c then show that, abc – (ab + bc + ca) + 3 (a + b + c) = 0

Sol. Let the given points lie on the line $lx + my + n = 0 \Rightarrow l\frac{t^3}{t-1} + m\frac{t^3-3}{t-1} + n = 0$

When t = a, b, c , this simplifies to $lt^3 + mt^2 + nt - (3m + n) = 0$

$\Rightarrow a + b + c = -\frac{m}{\ell}$; $ab + bc + ca = \frac{n}{\ell}$; $abc = \frac{3m+n}{\ell}$ $\Rightarrow$ result

Equation Of A Straight Line In Various Forms :

(i) **Slope – intercept form :** y = mx + c is the equation of a straight line whose slope is m & which makes an intercept c on the y–axis.

(ii) **Slope one point form :** $y - y_1 = m(x - x_1)$ is the equation of a straight line whose slope is m & which passes through the point (x_1, y_1).

(iii) **Parametric form :** The equation of the line in parametric form is given by

$\frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta}$ = r (say). Where 'r' is the distance of any point (x , y) on the line from the fixed point (x_1, y_1) on the line. r is positive if the point (x, y) is on the right of (x_1, y_1) and negative if (x, y) lies on the left of (x_1, y_1).

(iv) **Two point form :** $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ is the equation of a straight line which passes through the points (x_1, y_1) & (x_2, y_2).

(v) **Intercept form :** $\frac{x}{a} + \frac{y}{b} = 1$ is the equation of a straight line which makes intercepts a & b on OX & OY respectively.

(vi) **Perpendicular form :** $x\cos\alpha + y\sin\alpha = p$ is the equation of the straight line where the length of the perpendicular from the origin O on the line is p and this perpendicular makes angle α with positive side of x–axis.

(vii) **General Form :** ax + by + c = 0 is the equation of a straight line in the general form

Ex. 14 Find the equation of a line passing through (2, –3) and iclined at an angle of 135° with the postive direction x-axis.

Sol. Here m = slope of the line = tan 135° = tan (90° + 45°) = – cot 45° = –1, $x_1 = 2$, $y_1 = -3$

So, the equation of the line is $y - y_1 = m(x - x_1)$

i.e. $y - (-3) = -1(x - 2)$ or $y + 3 = -x + 2$ or $x + y + 1 = 0$ Ans.

Ex. 15 Find the equation of the line whish passes through the point (3, 4) and the sum of its intercepts on the axes is 14.

Sol. Let the equation of the line by $\frac{x}{a} + \frac{y}{b} = 1$ ...(i)

This passes through (3, 4), therefore $\frac{3}{a} + \frac{4}{b} = 1$ ...(ii)

It is given that a + b = 14 ⇒ 14 – a. Putting b = 14 – a in (ii), we get $\frac{3}{a} + \frac{4}{14-a} = 1$

$\Rightarrow \quad a^2 - 13a + 42 = 0$

$\Rightarrow \quad (a - 7)(a - 6) = 0 \Rightarrow a = 7, 6$

for a = 7, b = 14 – 7 = 7 and for a = 6, b = 14 – 6 = 8.

Putting the values of a and b in (i), we get the equations of the lines

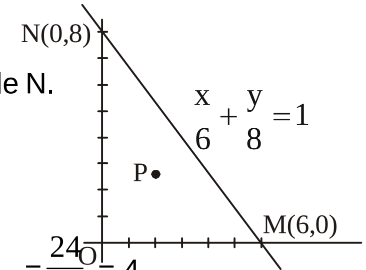
$\frac{x}{7} + \frac{y}{7} = 1$ and $\frac{y}{6} + \frac{y}{8} = 1$

or $\quad x + y = 7$ and $4x + 3y = 24$ Ans.

Ex. 16 Line $\frac{x}{6}+\frac{y}{8}=1$ intersects the x and y axes at M and N respectively. If the coordinates of the point P lying inside the triangle OMN (where 'O' is origin) are (a, b) such that the areas of the triangle POM, PON and PMN are equal. Find

(a) the coordinates of the point P and

(b) the radius of the circle escribed opposite to the angle N.



Sol. Note that 'P' is the centroid of Δ OMN $\Rightarrow$ P$\left(2, \frac{8}{3}\right)$

and $r_1 = \frac{\Delta}{(s-a)}$ where Δ = 24; s = $\frac{6+8+10}{2}$ = 12; a = 6 $= \frac{24}{6} = 4$

Ex. 17 Three straight lines l_1, l_2 and l_3 have slopes 1/2, 1/3 and 1/4 respectively. All three lines have the same y-intercept. If the sum of the x-intercept of three lines is 36 then find the y-intercept.

Sol. $l_1 : y = \frac{1}{2}x + c \quad \Rightarrow \quad$ x-intercept is – 2c

$l_2 : y = \frac{1}{3}x + c \quad \Rightarrow \quad$ x-intercept is – 3c

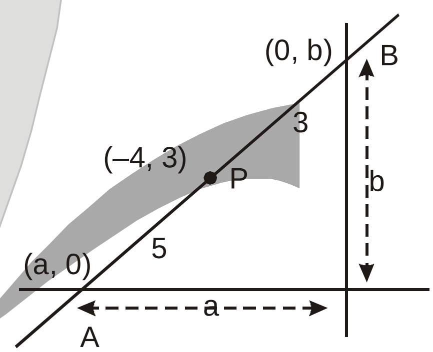
$l_3 : y = \frac{1}{4}x + c \quad \Rightarrow \quad$ x-intercept is – 4c

$\therefore$ – 2c – 3c – 4c = 36 $\Rightarrow$ – 9c = 36 $\Rightarrow$ c = – 4

Ex. 18 Find the equation to the straight line which passes through the point (–4, 3) and is such that the portion of it between the axes is divided by the point in the ratio 5 : 3.

Sol. Let the required straight line cuts the axes of x and y at A(a, 0) and B (0, b) respectively. Hence the co-ordinates of the point P which divides AB in the ratio of 5 : 3 are given by

$x = \frac{3a}{8}$ and $y = \frac{5b}{8}$.



By hypothesis, co-ordinates of P are (–4, 3).

Hence $\frac{3a}{8}$ = –4 and $\frac{5b}{8}$ = 3 or a = $\frac{32}{-3}$ and b = $\frac{24}{5}$.

$\therefore$ The required equation is $\frac{x}{-\frac{32}{3}} + \frac{y}{\frac{24}{5}} = 1$

or 20y – 9x = 96.

Ex. 19 Find the co-ordinates of the points of intersection of the straight lines, whose equations are x cos ϕ_1 + y sin ϕ_1 = a and x cos ϕ_2 + y sin ϕ_2 = a.

Sol. The equation are

$x\cos\phi_1 + y\sin\phi_1 - a = 0$ ...(1)

and $x\cos\phi_2 + y\sin\phi_2 - a = 0$. ...(2) By cross–multiplication

or $\frac{x}{a(\sin\phi_2 - \sin\phi_1)} = \frac{y}{a(\cos\phi_1 - \cos\phi_2)} = \frac{1}{\sin(\phi_2 - \phi_1)}$

or $\frac{x}{a2.\cos\frac{1}{2}(\phi_2+\phi_1)\sin\frac{1}{2}(\phi_2-\phi_1)} = \frac{y}{a2.\sin\frac{1}{2}(\phi_1+\phi_2)\sin\frac{1}{2}(\phi_2-\phi_1)}$

$$= \frac{1}{2.\sin\frac{1}{2}(\phi_2+\phi_1)\cos\frac{1}{2}(\phi_2-\phi_1)}$$

$$x = \frac{a\cos\frac{1}{2}(\phi_2+\phi_1)}{\cos\frac{1}{2}(\phi_2-\phi_1)}, \quad y = \frac{a\sin\frac{1}{2}(\phi_1+\phi_2)}{\cos\frac{1}{2}(\phi_1-\phi_2)}$$

Ex. 20 Show that the centroid of the triangle of which the three altitudes to its sides lie on the line $y = m_1 x$; $y = m_2 x$ & $y = m_3 x$ lie on the line ,

$$y\,(m_1 m_2 + m_2 m_3 + m_3 m_1 + 3) = (m_1 + m_2 + m_3 + 3\,m_1 m_2 m_3)\,x.$$

Sol. Equation of BC $y = -\frac{1}{m_1}\,x + c$

Solve with $y = m_2 x$ & $y = m_3 x$ to get the co-ordinates of B and C. Now equation of AC

with slope $-\frac{1}{m_2}$ and passing through C can be

known. Solve it with $y = m_1 x$ to get A. Now compute h & k $\Rightarrow y = \frac{k}{h}\,x$

Ex. 21 The opposite angular points of a square are (3, 4) and (1, –1). Find the coordinates of the other two vertices.

Sol. Slope of AC = $\frac{5}{2}$; Slope of BD = $-\frac{2}{5} = \tan\theta$

$\therefore \quad \cos\theta = -\frac{5}{\sqrt{29}}, \quad \sin\theta = \frac{2}{\sqrt{29}}$

Length of AC = $\sqrt{29}$ = BD and mid–point E of AC is $\left(2, \frac{3}{2}\right)$. Any line through E is

$$\frac{x-2}{\cos\theta} = \frac{y - 3/2}{\sin\theta} = r, -r \qquad \text{where } r = \frac{1}{2}\sqrt{29}$$

$x = r\cos\theta + 2,\ y = r\sin\theta + 3/2$

$x = \frac{1}{2}\sqrt{29}\left(-\frac{5}{\sqrt{29}}\right) + 2,\ y = \frac{1}{2}\sqrt{29}, \frac{2}{\sqrt{29}} + \frac{3}{2} \qquad B = (x, y) = \left(-\frac{1}{2}, \frac{5}{2}\right).$

Writting – r for r in above, the point D is (9/2, 1/2)

Ex. 22 Find the equation of the line through the point A(2, 3) and making an angle of 45° with the x-axis. Also determine the length of intercept on it between A and the line x + y + 1 = 0.

Sol. The equation of a line through A and making an angle of 45° with the x-axis is

$$\frac{x-2}{\cos 45^\circ} = \frac{y-3}{\sin 45^\circ} \text{ or } \frac{x-2}{\frac{1}{\sqrt{2}}} = \frac{y-3}{\frac{1}{\sqrt{2}}}$$

or $\quad x - y + 1 = 0$

Suppose this line meets the line x + y + 1 = 0 at P such that AP = r. Then the coordinates of the P are given by

$$\frac{x-2}{\cos 45^\circ} = \frac{y-3}{\sin 45^\circ} = r \Rightarrow \qquad x = 2 + r\cos 45^\circ,\ y = 3 + r\sin 45^\circ$$

$\Rightarrow \quad x = 2 + \frac{r}{\sqrt{2}},\ y = 3 + \frac{r}{\sqrt{2}}$

Thus the coordinates of P are $\left(2 + \frac{r}{\sqrt{2}}, 3 + \frac{r}{\sqrt{2}}\right)$

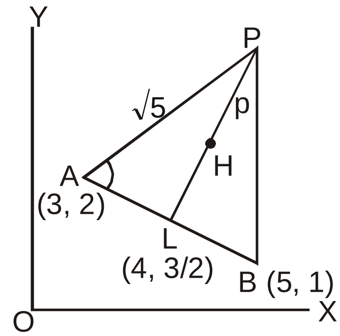
Since P lies on x + y + 1 = 0, so $2 + 2 + \frac{r}{\sqrt{2}}, 3 + \frac{r}{\sqrt{2}} + 1 = 0$

$\Rightarrow \quad \sqrt{2}\,r = -6 \Rightarrow r = -3\sqrt{2} \qquad \Rightarrow \qquad$ length AP = $|\,r\,| = 3\sqrt{2}$

Thus, the length of the intercept = $3\sqrt{2}$ Ans.

Ex. 23 A and B are two fixed points whose co-ordinates respectively are (3, 2) and (5, 1). ABP is an equilateral triangle on AB situated on the side opposite to that of origin. Find the co-ordinates of P and those of the orthocentre of triangle ABP.

Sol. Equation of AB is x + 2y = 7 and its length a = $\sqrt{5}$, and mid-point of AB is the point L(4, 3/2). If P be the vertex of the equilateral triangle then its perpendicular distance p from AB is a sin 60° or p = $\sqrt{5}$, ($\sqrt{3}$,/2) = 1/2 $\sqrt{15}$



Also distance of orthocentre H from AB is 1/3 p.
Now both H and P lie on a line perpendicular to AB whose slope will be 2 and passing through L(4, 3/2),

$\therefore$ tan θ = 2 or sin θ = 2 / $\sqrt{5}$ and cos θ = 1/ $\sqrt{5}$

Hence H and P lie on $\dfrac{x-4}{\cos\theta} = \dfrac{y-3/2}{\sin\theta}$ = p for P

and = 1/3p for H.

x = 4 + p cos θ, y = 3/2 + p sin θ for P

x = 4 + (p/3) cos θ, y = 3/2 + (p/3) sin θ for H

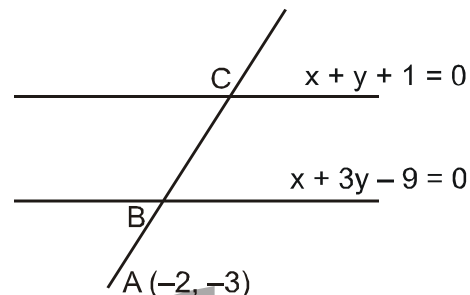
Putting the values fo p, cos θ and sin θ in the above, we get

Point P(4 + $\sqrt{3}$/2, 3/2 + $\sqrt{3}$) Point H[5 + $\sqrt{3}$/6, 3/2 + (1/2)$\sqrt{3}$

Ex. 24 A straight line through A (– 2, – 3) cuts the lines x + 3y – 9 = 0 & x + y + 1 = 0 at B and C respectively. Find the equation of the line if AB. AC = 20.

[Ans. : x – y = 1 & 3x – y + 3 = 0]

Sol. $\dfrac{x+2}{\cos\theta} = \dfrac{y+3}{\sin\theta}$ = r



Now for r = r_1 and r = r_2

r_1 cos θ – 2 + r_1 sin θ – 3 + 1 = 0

r_1 (cos θ + sin θ) = 4

Similarly r_2 (cos θ + 3 sin θ) = 20

Given $r_1 r_2$ = 20 $\Rightarrow \dfrac{80}{(\cos\theta+\sin\theta)\,(\cos\theta+3\sin\theta)}$ = 20

Multiplying tan θ = 3 or 1

Hence the equation of the line , y + 3 = 3 (x + 2) or y + 3 = (x + 2)

Ex. 25 A line is drawn throuhg a variable point A (t + 1, 2t) so as to meet the following lines in points indicated with them : 7x + y – 16 = 0 in B, 5x – y – 8 = 0 in C, x – 5y + 8 = 0 in D.
Show that AC, AB, AD are in H.P.

Sol. Any line through A (t + 1, 2t) is

$$\frac{x-(t-1)}{\cos\theta} = \frac{y-2t}{\sin\theta} = \begin{matrix} r_1, & r_2, & r_3 \\ B & C & D \end{matrix}$$ where r_1 = AB, r_2 = AC, r_3 = AD

[r_1 cos θ + (t + 1), r_1 sin θ + 2t] is point B which lies on 7x + y – 16 = 0

then $r_1 = \dfrac{9(1-t)}{7\cos\theta\sin\theta}$ ...(1)

Similarly $r_2 = \dfrac{3(1-t)}{5\cos\theta-\sin\theta}$ ...(2)

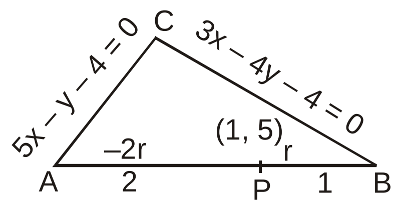
and $r_3 = \dfrac{9(1-t)}{5\sin\theta-\cos\theta}$ ...(3)

Now we have to show that r_2, r_1, r_3 are in H.P. or $\dfrac{2}{r_1} = \dfrac{1}{r_2} + \dfrac{1}{r_3}$

Now $\dfrac{1}{r_2} + \dfrac{1}{r_3} = \dfrac{5\cos\theta-\sin\theta}{3(1-t)} + \dfrac{5\sin\theta-\cos\theta}{9(1-t)} = \dfrac{14\cos\theta+2\sin\theta}{9(1-t)} = \dfrac{2(7\cos\theta+\sin\theta)}{9(1-t)}$

$= \dfrac{2}{r_1}$ by (1). Hence in H.P.

Ex. 26 The base of a triangle ABC passes through the point P (1, 5) which divides it in the ratio 2 : 1. If the equation of the sides are AC, $5x - y - 4 = 0$ and BC, $3x - 4y - 4 = 0$, then the co-ordinates of vertex A are $\left(\frac{75}{17}, \frac{307}{17}\right)$.



Sol. Any line through P(1, 5) is $\frac{x-1}{\cos\theta} = \frac{y-5}{\sin\theta} = \underset{\text{for B}}{r}, \underset{\text{for A}}{-2r}$

A $(1 - 2r\cos\theta, 5 - 2r\sin\theta)$ B $(1 + r\cos\theta, 5 = r\sin\theta)$

A lies on $5x - y - 4 = 0$ $5(1 - 2r\cos\theta) - (5 - 2r\sin\theta) - 4 = 0$

or $r(\sin\theta - 5\cos\theta) = 2$ ....(1)

B lies on $3x - 4y - 4 = 0$ $3(1 + r\cos\theta) - 4(5 + r\sin\theta) - 4 = 0$

or $r(3\cos\theta - 4\sin\theta) = 21$ ...(2)

Dividing (1) by (2) to eliminate r $\frac{\sin\theta - 5\cos\theta}{3\cos\theta - 4\sin\theta} = \frac{2}{21}$

$\therefore$ $29\sin\theta = 111\cos\theta$ $\therefore$ $\frac{\sin\theta}{111} = \frac{\cos\theta}{29} = k$ say

Hence from (1) $r(111k - 145k) = 2$ or $rk = -1/17$

Same value will be found from (2)

If point A is (x, y) then $x = 1 - 58\left(-\frac{1}{17}\right) = \frac{75}{17}$

$y = 5 - 2r(111k) = 5 - 222\left(-\frac{1}{17}\right) = \frac{307}{17}$ $\therefore$ A is $\left(\frac{75}{17}, \frac{307}{17}\right)$ which satisfies $5x - y - 4 = 0$.

F. POSITION OF A GIVEN POINT RELATIVE TO A GIVEN LINE

The fig. shows a point $P(x_1, y_1)$ lying above a given line

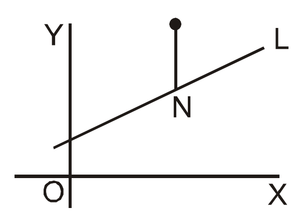
$L(x, y) \equiv ax + by + c = 0$ ...(1)

If an ordinate is dropped from P to meet the line L at N, then the x coordinate of N will be x_1. Putting $x = x_1$ in the equation (1) gives

y coordinate of N = $\frac{(ax_1 + c)}{b}$

If $P(x_1, y_1)$ lies above the lien, then we have

$$y_1 > -\frac{(ax_1 + c)}{b}$$



i.e., $y_1 + \frac{(ax_1 + c)}{b} > 0$

i.e. $\frac{(ax_1 + by_1 + c)}{b} > 0$

i.e. $\frac{L(x_1, y_1)}{b} > 0$ ...(2)

Hence, if $P(x_1, y_1)$ satisfies equation (2), it would mean that P lies above the line $ax + by + c = 0$, and if

$\frac{L(x_1, y_1)}{b} < 0$ ...(3)

it would mean that P lies below the line $ax + by + c = 0$.

The Ratio In Which A Given Line Divides The Line Segment Joining Two Points

Let the given line $ax + by + c = 0$ divide the line segment joining $A(x_1, y_1)$ & $B(x_2, y_2)$ in the ratio m : n, then $\frac{m}{n} = -\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}$. If A & B are on the same side of the given line then $\frac{m}{n}$ is negative but if A & B are on opposite sides of the given line , then $\frac{m}{n}$ is positive

Position Of The Point (x_1, y_1) Relative To The Line $ax + by + c = 0$:

If $ax_1 + by_1 + c$ is of the same sign as c, then the point (x_1, y_1) lie on the origin side of $ax + by + c = 0$. But if the sign of $ax_1 + by_1 + c$ is opposite to that of c, the point (x_1, y_1) will lie on the non–origin side of $ax + by + c = 0$.

Ex. 27 Show that (1, 4) and (0, –3) lie on the opposite sides of the line $x + 3y + 7 = 0$.

Sol. At (1, 4), the value of $x + 3y + 7 = 1 + 3(4) + 7 = 20 > 0$.

At (0, – 3), the value of $x + 3y + 7 = 0 + 3(–3) + 7 = –2 < 0$

$\therefore$ The points (1, 4) and (0, –3) are on the opposite sides of the given line. Ans.

Ex. 28 Find the ratio in which the line joining the point A(1, 2) and B (–3, 4) is divided by the line $x + y – 5 = 0$.

Sol. Let the line $x + y = 5$ divides AB in the ratio k : 1 and P

$\therefore$ coordinate of P are $\left(\frac{-3k+1}{k+1}, \frac{4k+2}{k+1}\right)$

Since P lies on $x + y – 5 = 0$

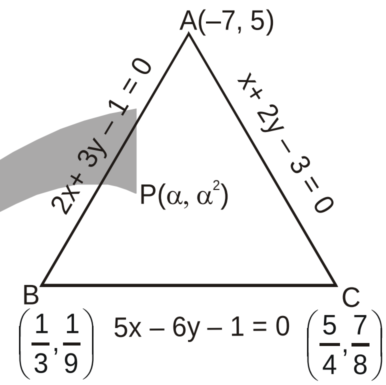
$\therefore$ $\frac{-3k+1}{k+1} + \frac{4k+2}{k+1} - 5 = 0$ $\Rightarrow$ $k = -\frac{1}{2}$

$\therefore$ Required ratio is 1 : 2 externally Ans.

Ex. 29 Determine all values of α for which the point (α, α^2) lies inside the triangle formed by the lines $2x + 3y – 1 = 0$, $x + 2y – 3 = 0$, $5x – 6y – 1 = 0$.

Sol. Solving equations of the lines two at a time we get the vertices of the given triangle as A (–7, 5), B (1/3, 1/9) and C (5/4, 7/8)

Let $P(\alpha, \alpha^2)$ be a point inside the triangle ABC (fig.). Since A and P lie on the same side of the line $5x – 6y – 1 = 0$, both $5 – (7) – 6(5) – 1$ and $5\alpha – 6\alpha^2 – 1$ must have the same sign.



$\Rightarrow$ $5\alpha – 6\alpha^2 – 1 < 0$ or $6\alpha^2 – 5\alpha + 1 > 0$

$\Rightarrow$ $(3\alpha – 1)(2\alpha – 1) > 0$

$\Rightarrow$ Either $\alpha < 1/3$ or $\alpha > 1/2$ ...(1)

Again since B and P lie on the same side of the line $x + 2y – 3 = 0$,

$(1/3) + (2/9)$ and $\alpha + 2\alpha^2 – 3$ have the same sign.

$\Rightarrow$ $2\alpha^2 + \alpha – 3 < 0$ $\Rightarrow$ $(2\alpha + 3)(\alpha – 1) < 0$

$\Rightarrow$ $–3/2 < \alpha < 1$ ...(2)

Lastly since C and P lie on the same side of the line $2x + 3y – 1 = 0$.

$2 \times (5/4) + 3 \times (7/8) – 1$ and $(2\alpha + 3\alpha^2 – 1$ have the same sign

$\Rightarrow$ $3\alpha^2 + 2\alpha – 1 > 0$ $\Rightarrow (3\alpha – 1)(\alpha + 1) > 0$

$\Rightarrow$ $\alpha < –1$ or $\alpha > 1/3$ ...(3)

Now (1), (2), (3) hold simultaneously if $–3/2 < \alpha < –1$ or $1/2 < \alpha < 1$.

G. LENGTH OF PERPENDICULAR FROM A POINT ON A LINE

The length of perpendicular from $P(x_1, y_1)$ on $ax + by + c = 0$ is $\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}}\right|$.

Reflection of point about a line :

(i) Foot of the perpendicular from a point on the line is

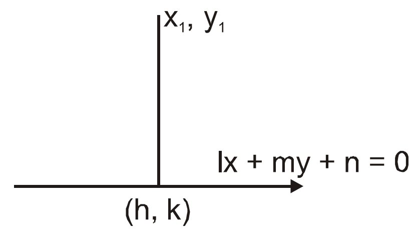
$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{ax_1 + by_1 + c}{a^2 + b^2}$$

(ii) The image of a poit (x_1, y_1) about the line $ax + by + c = 0$ is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = 2\frac{ax_1 + by_1 + c}{a^2 + b^2}$$

Ex. 30 If (h, k) be the foot of the perpendicular from (x_1, y_1) on $lx + my + n = 0$, show that

$$\frac{h - x_1}{l} = \frac{k - y_1}{m} = -\frac{lx_1 + my_1 + n}{l^2 + m^2}.$$



Sol. $\frac{k - y_1}{h - x_1}\left(-\frac{l}{m}\right) = -1$

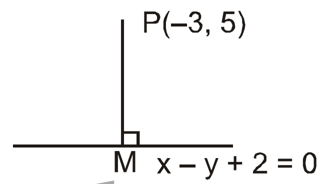
$$\Rightarrow \frac{h - x_1}{l} = \frac{k - y_1}{m}$$

or $\frac{hl - lx_1}{l^2} = \frac{km - mx_1}{m^2} = \frac{hl - lx_1 + km - my_1}{l^2 + m^2}$

or $\frac{h - x_1}{l} = \frac{k - y_1}{m} = \frac{lx_1 - my_1 - n}{l^2 + m^2}$

Ex. 31 Find the foot of perpendicular of the line drawn from P(–3, 5) on the line $x - y + 2 = 0$

Sol. Slope of PM = –1



$\therefore$ Equation of PM is $x + y - 2 = 0$ ....(i)

Solving equation (i) with $x - y + 2 = 0$, we get coordinates of M(0, 2) Ans.

Alter

Here, $\frac{x + 3}{1} = \frac{y - 5}{-1} = -\frac{(1 \times (-3) + (-1) \times 5 + 2)}{(1)^2 + (-1)^2}$

$\Rightarrow \frac{x + 3}{1} = \frac{y - 5}{-1} = 3$ $\Rightarrow$ $x + 3 = 3$ $\Rightarrow$ $x = 0$

and $y - 5 = -3$ $\Rightarrow$ $y = 2$

$\therefore$ M is(0, 2) Ans.

Ex. 32 Find the image of the point P(–1, 2) in the line mirror $2x - 3y + 4 = 0$.

Sol. The image of P(–1, 2) about the line $2x - 3y + 4 = 0$ is

$$\frac{x + 1}{2} = \frac{y - 2}{-3} = -2\frac{[2(-1) - 3(2) + 4]}{2^2 + (-3)^2} \quad ; \quad \frac{x + 1}{2} = \frac{y - 2}{-3} = \frac{8}{13}$$

$\Rightarrow$ $13x + 13 = 16$ $\Rightarrow$ $x = \frac{3}{13}$

& $13y - 26 = -24$ $\Rightarrow$ $y = \frac{2}{13}$

$\therefore$ image is $\left(\frac{3}{13}, \frac{2}{13}\right)$ Ans.

Ex. 33 Find all points on x + y = 4 that lie at a unit distance from the line 4x + 3y – 10 = 0.

Sol. Note that the coordinates of an arbitrary point on x + y = 4 can be obtained by putting x + t (or y = t) and then obtaining y (or x) from the equation of the line, where t is a parameter.
Putting x = t in the equation x + y = 4 of the given, we obtain y = 4 – t. So, coordinates of an arbitrary point on the given line are P(t, 4 – t), Let P(t, 4 – t) be the required point. Then, distance of P from the line 4x + 3y – 10 = 0 is unity i.e.

$$\Rightarrow \quad \left|\frac{4t+3(4-t)-10}{\sqrt{4^2+3^2}}\right| = 1 \Rightarrow |t+2| = 5 \Rightarrow t+2 = \pm 5$$

$\Rightarrow \quad t = -7$ or $t = 3$

Hence, required points are (–7, 11) and (3, 1) Ans.

Ex. 34 On the straight line y = x + 2, find the point for which the sum of the squared distances from the straight line 3x – 4y + 8 = 0 and 3x – y – 1 = 0 would be the least possible.

Sol. Point be (x, y) but it lies on y = x + 2

So (x, x + 2) $\quad F(x) = \left[\frac{3x-4(x+2)+8}{\sqrt{3^2+4^2}}\right]^2 + \left[\frac{3x-(x+2)-1}{\sqrt{3^2+1^2}}\right]^2$

$$= \frac{2x^2+5[4x^2-12x+9]}{50} \qquad = \frac{22\left[\left(x-\frac{30}{22}\right)^2-\frac{900}{484}\right]+45}{50}$$

F(x) is minimum at $x = \frac{15}{11}$. So point is $\left(\frac{15}{11}, \frac{37}{11}\right)$

Ex. 35 If p and p′ be the perpediculars from the origin upon the straight lines whose equations are x sec θ + y cosec θ = a and x cos θ – y sin θ = a cos 2θ, prove that $4p^2 + p'^2 = a^2$.

Sol. The point is given to be the origin (0, 0) and lines are

x sec θ + y cosec θ = a ...(1)

and x cos θ – y sin θ = a cos 2θ. ...(2)

If p and p′ are the perpendiculars from (0, 0) on (1) and (2) respectively, then

$$p = \left|\frac{0.\sec\theta + 0.\operatorname{cosec}\theta - a}{\sqrt{(\sec^2\theta + \operatorname{cosec}^2\theta)}}\right| = \frac{a}{\sqrt{(\sec^2\theta + \operatorname{cosec}^2\theta)}} \qquad ...(3)$$

and $$p' = \left|\frac{0.\cos\theta - 0.\sin\theta - a\cos 2\theta}{\sqrt{(\cos^2\theta + \sin^2\theta)}}\right| = \left|\frac{a\cos 2\theta}{1}\right| \qquad ...(4)$$

Now, $4p^2 + p'^2 = \frac{4a^2}{\sec^2\theta + \operatorname{cosec}^2\theta} + a^2\cos^2 2\theta$ (putting the values of p and p′)

$$= \frac{4a^2\cos^2\theta\sin^2\theta}{\sin^2\theta+\cos^2\theta} + a^2\cos^2 2\theta \qquad = a^2\sin^2 2\theta + a^2\cos^2 2\theta = a^2.$$ Proved.

H. ANGLE BETWEEN TWO STRAIGHT LINES IN TERMS OF THEIR SLOPES

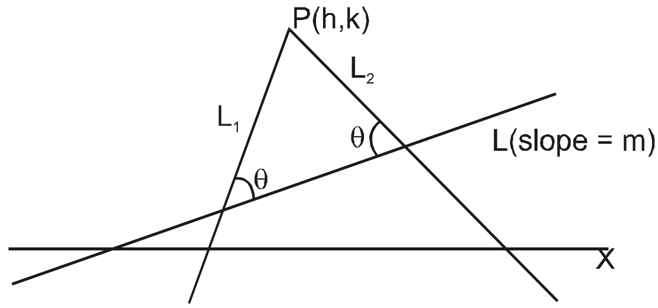
If m_1 & m_2 are the slopes of two intersecting straight lines ($m_1 m_2 \neq -1$) & θ is the acute angle between them, then $\tan\theta = \left|\frac{m_1 - m_2}{1+m_1 m_2}\right|$.

Note : Let m_1, m_2, m_3 are the slopes of three lines $L_1 = 0$; $L_2 = 0$; $L_3 = 0$ where $m_1 > m_2 > m_3$ then the interior angles of the Δ ABC found by these lines are given by,

$$\tan A = \frac{m_1 - m_2}{1 + m_1 m_2} \ ; \ \tan B = \frac{m_2 - m_3}{1 + m_2 m_3} \ \& \ \tan C = \frac{m_3 - m_1}{1 + m_3 m_1}$$

Equation of Line through P(h, k) and Inclined at Angle θ to the Line $L \equiv y - mx - c = 0$

There will be two such lines L_1, L_2 satisfying the given condition (see fig.). To find their equations, we only need to know their slopes.



Let m′ represent the slope of either of these lines, the by Art. 4.3, m′ must satisfy the following condition

$$\pm \tan\theta = \frac{m' - m}{1 + mm'} \qquad ...(1)$$

Solving equation (1), we have

$$m' = \frac{m + \tan\theta}{1 - m\tan\theta}, \frac{m - \tan\theta}{1 + m\tan\theta}$$

Hence, by Art. 4.1.2. equations of the required lines are

$$y - k = \frac{m + \tan\theta}{1 - m\tan\theta}(x - h) \qquad ...(2)$$

and $$y - k = \frac{m - \tan\theta}{1 + m\tan\theta}(x - h) \qquad ...(3)$$

Ex. 36 The acute angle between two lines is $\frac{\pi}{4}$ and slope of one of them is 1/2. Find the slope of the other line.

Sol. If q be the acute angle between thelines with slopes m_1 and m_2, then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

Let $\theta = \frac{\pi}{4}$ and $m_1 = \frac{1}{2}$

$$\therefore \quad \tan\frac{\pi}{4} = \left|\frac{\frac{1}{2} - m_2}{1 + \frac{1}{2}m_2}\right| \quad \Rightarrow \quad 1 = \left|\frac{1 - 2m_2}{2 + m_2}\right| \quad \Rightarrow \quad \frac{1 - 2m_2}{2 + m_2} = +1 \text{ or } -1$$

Now $\frac{1 - 2m_2}{2 + m_2} = 1 \quad \Rightarrow \quad m_2 = -\frac{1}{3}$ and $\frac{1 - 2m_2}{2 + m_2} = 1 \quad \Rightarrow \quad m_2 = 3.$

$\therefore$ The slope of the other line is either –1/3 or 3 Ans.

Ex. 37 Find the equation of the straight line which passes through the origin and making angle 60° with the line $x + \sqrt{3}\, y + 3\sqrt{3} = 0$.

Sol. Given line is $x + \sqrt{3}\, y + 3\sqrt{3} = 0$.

$$\Rightarrow \quad y = \left(-\frac{1}{\sqrt{3}}\right)x - 3 \qquad \therefore \quad \text{Slope of (1)} = -\frac{1}{\sqrt{3}}$$

Let slope of the required line be m. Also between these lines is given to be 60°

$\Rightarrow \quad \tan 60° \left| \dfrac{m-\left(-\dfrac{1}{\sqrt{3}}\right)}{1+\left(-\dfrac{1}{\sqrt{3}}\right)} \right| \Rightarrow \quad \sqrt{3} = \left| \dfrac{\sqrt{3}m+1}{\sqrt{3}-m} \right| \quad \Rightarrow \quad \dfrac{\sqrt{3}m+1}{\sqrt{3}-m} = \pm\sqrt{3}$

$\dfrac{\sqrt{3}m+1}{\sqrt{3}-m} = -\sqrt{3} \Rightarrow \sqrt{3}\,m + 1 = 3 - \sqrt{3}\,m \Rightarrow \quad m = \dfrac{1}{\sqrt{3}}$

Using y = mx + c, the equation of the required line is

$y = \dfrac{1}{\sqrt{3}}x + 0$ i.e. $\quad x - \sqrt{3}\,y = 0 \qquad$ ($\because$ This passes through origin, so c = 0)

$\dfrac{\sqrt{3}m+1}{\sqrt{3}-m} = -\sqrt{3} \qquad \Rightarrow \qquad \sqrt{3}\,m + 1 = -3 + \sqrt{3}\,m$

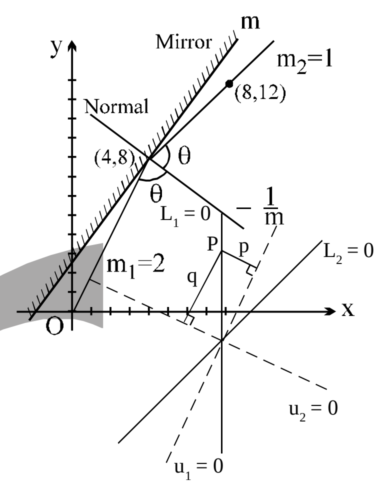
$\Rightarrow$ m is not defined

$\therefore$ The slope of the required lien is not defined. Thus, the required lien is a vertical line. This line is to pass through the origin.

$\therefore$ the equation of the required line is x = 0 Ans.

Ex. 38 Starting at the origin, a beam of light hits a mirror (in the form of a line) at the point A(4, 8) and is reflected at the point B(8, 12). Compute the slope of the mirror.

Sol. Let the slope of the line mirror is *m*. Hence slope of normal is – 1/m



Equating the two values of θ, we get

$\dfrac{-\dfrac{1}{m}-2}{1-\dfrac{2}{m}} = \dfrac{1+\dfrac{1}{m}}{1-\dfrac{1}{m}} \qquad \Rightarrow \qquad -\dfrac{2m+1}{m-2} = \left(\dfrac{m+1}{m-1}\right)$

$3m^2 - 2m - 3 = 0$

$\therefore \quad m = \dfrac{1+\sqrt{10}}{3} \quad$ or $\quad m = \dfrac{1-\sqrt{10}}{3}$ which is rejected (think !)

[$m \in (1, 2)$]

$\therefore$ slope of the mirror is $\dfrac{1+\sqrt{10}}{3}$

Ex. 39 Show that the equations to the straight lines passing through the point (3, –2) and inclined at 60° to the line √3x + y = 1 are y + 2 = 0 and y – √3x + 2 + 3√3 = 0.

Sol. The equation to the straight line passing through (3, –2) and inclined at an angle of 60° to the line $\sqrt{3}x + y = 1$ ...(1)

are (a) $\quad y + 2 = \dfrac{-\sqrt{3}+\tan 60°}{1+\sqrt{3}\tan 60°}(x-3)$ ($\because$ m of the given line is $-\sqrt{3}$)

or $\quad y + 2 = \dfrac{-\sqrt{3}+\sqrt{3}}{1-(\sqrt{3})\sqrt{3}}(x-3)$ or y + 2 = 0.

(b) $\quad y + 2 = \dfrac{-\sqrt{3}-\tan 60°}{1+(\sqrt{3})\tan 60°}(x-3)$

or $\quad y + 2 = \dfrac{-\sqrt{3}-\sqrt{3}}{1-\sqrt{3}.\sqrt{3}}(x-3) \quad$ or $\quad y + 2 = \dfrac{2\sqrt{3}}{2}(x-3)$ or $y - \sqrt{3}x + 2\ 3\sqrt{3} = 0$.

I. PERPENDICULAR / PARALLEL LINES

Perpendicular Lines

(i) When two lines of slopes m_1 & m_2 are at right angles, the product of their slopes is -1, i.e. $m_1 m_2 = -1$. Thus any line perpendicular to $ax + by + c = 0$ is of the form $bx - ay + k = 0$, where k is any parameter.

(ii) Straight lines $ax + by + c = 0$ & $a'x + b'y + c' = 0$ are at right angles if & only if $aa' + bb' = 0$.

Parallel Lines

(i) When two straight lines are parallel their slopes are equal. Thus any line parallel to $y = mx + c$ is of the type $y = mx + d$, where k is a parameter.

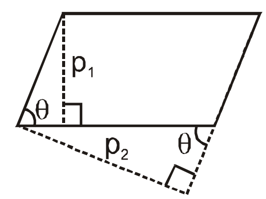
(ii) Two lines $ax + by + c = 0$ and $a'x + b'y + c' =$ - are parallel if $\frac{a}{a'} = \frac{b}{b'} \neq \frac{c}{c'}$.

Thus any line parallel to $ax + by + c = 0$ is of the type $ax + by + k = 0$, where k is a parameter

(iii) The distance between two parallel lines with equations $ax + by + c_1 = 0$ & $ax + by + c_2 = 0$ is $\left|\frac{c_1 - c_2}{\sqrt{a^2 + b^2}}\right|$

Note : Coefficients of x & y in both the equations must be same.

(iv) The area of the parallelogram $= \frac{p_1 p_2}{\sin\theta}$, where p_1 & p_2 are distances between two pairs of opposite sides & θ is the angle between any two adjacent sides. Note that area of the parallelogram bounded by the lines $y = m_1x + c_1$, $y = m_1x + c_2$ and $y = m_2x + d_1$, $y = m_2x + d_2$ is given by $\left|\frac{(c_1 - c_2)(d_1 - d_2)}{m_1 - m_2}\right|$.



Ex. 40 The equations of the two sides of a rhombous are $3x - 10y + 37 = 0$ and $9x + 2y - 17 = 0$ and the equation of one of its diagonals is $3x - 2y - 19 = 0$. Find the equations of two other sides of the rhombous and the equation to its second diagonal.

Sol. equation of BD is $3x - 2y - 19 = 0$

AC will be perpendicular to BC and passing through (1, 4)

$\Rightarrow$ equation of AC $= 2x + 3y - 14 = 0$

other two sides are $9x + 2y - 113 = 0$ &

$3x - 10y - 59 = 0$

Ex. 41 Two sides of a square lie on the line $x + y = 1$ and $x + y + 2 = 0$. What is its area ?

Sol. Clearly the length of the side of the square is equal to the distance between the parallel lines

$x + y - 1 = 0$ ...(i) and $x + y + 2 = 0$ ...(ii)

Putting $x = 0$ in (i), we get $y = 1$. So (0, 1) is a point on line (i)

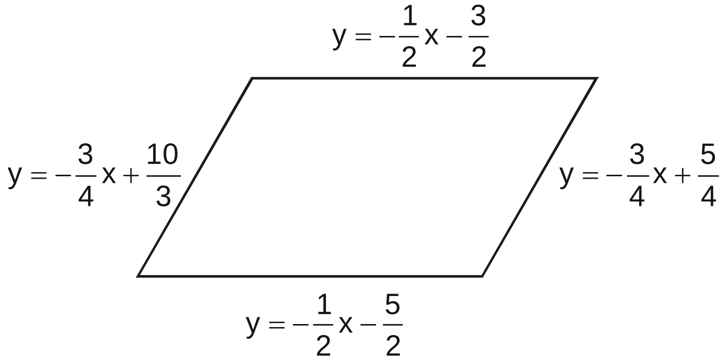
Now, Distance between the parallel lines

$= $ length of the $\perp$ from (0, 1) to $x + y + 2 = 0 = \frac{|0 + 1 + 2|}{\sqrt{1^2 + 1^2}} = \frac{3}{\sqrt{2}}$

Thus, the length of the side of the square is $\frac{3}{\sqrt{2}}$ and hence its area $= \left(\frac{3}{\sqrt{2}}\right)^2 = \frac{9}{2}$

Ex. 42 Find the area of the parallelogram whose sides are $x + 2y + 3 = 0$, $3x + 4y - 5 = 0$, $2x + 4y + 5 = 0$ and $3x + 4y - 10 = 0$

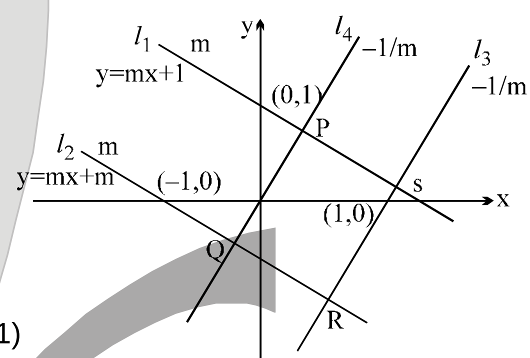
Sol.



Here, $c_1 = -\frac{3}{2}$, $c_2 = \frac{5}{2}$, $d_1 = \frac{10}{3}$, $d_2 = -\frac{5}{2}$, $m_1 = -\frac{1}{2}$, $m_2 = -\frac{3}{4}$

$$\therefore \quad \text{Area} = \left|\frac{\left(-\frac{3}{2}+\frac{5}{2}\right)\left(\frac{10}{3}+\frac{5}{2}\right)}{\left(-\frac{1}{2}+\frac{3}{4}\right)}\right| = \frac{70}{3} \text{ sq. units} \quad \text{Ans.}$$

Ex. 43 Two parallel lines pass through the point (0, 1) and (–1, 0) respectively. Two other lines are drawn through (1, 0) and (0, 0) respectively each perpendicular to the first two. The two sets of parallel lines intersect in four points that are the vertices of a square. Find all possible equations for the first two lines.



Sol. Equation of l_3 is $y = -\frac{1}{m}(x - 1)$ $\quad x + my - 1 = 0$

PQRS is a square

$\Rightarrow$ distance between l_1 and l_2 = distance between l_3 and l_4

$$\left|\frac{m-1}{\sqrt{1+m^2}}\right| = \left|\frac{1}{\sqrt{1+m^2}}\right|$$

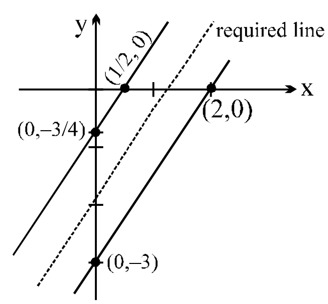
$\Rightarrow \quad |m - 1| = 1$. Thus $m = 0$ or $m = 2$

if $m = 2$, equation of lines are $y = 2x + 1$ and $y = 2(x + 1)$

if $m = 0$, lines are $y = 0$ and $y = 1$

Ex. 44 Find the equation of the line such that its distance from the lines $3x - 2y - 6 = 0$ and $6x - 4y - 3 = 0$ is equal.

Sol. Note that lines are parallel.



y intercept of the required line is $= -\frac{1}{2}\left(3+\frac{3}{4}\right) = -\frac{15}{8}$

Its slope is 3/2

equation is $y = \frac{3}{2}x - \frac{15}{8} \Rightarrow \quad 12x - 8y = 15$

J. CONCURRENCY OF LINES

Three lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ & $a_3x + b_3y + c_3 = 0$

are concurrent if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

Alternatively : If three constants A, B & C can be found such that $A(a_1x + b_1y + c_1) + B(a_2x + b_2y + c_2) + C(a_3x + b_3y + c_3) \equiv 0$, then the three straight lines are concurrent.

Ex. 45 Prove that the straight lines 4x + 7y = 9, 5x – 8y + 15 = 0 and 9x – y + 6 are concurrent.

Sol. Given lines are

$4x + 7y - 9 = 0$ ...(1)

$5x - 8y + 15 = 0$ ...(2)

and $9x - y + 6 = 0$ ...(3)

$$\Delta = \begin{vmatrix} 4 & 7 & -9 \\ 5 & -8 & 15 \\ 9 & -1 & 6 \end{vmatrix} = (4(-48 + 15) - 7(30 - 135) - 9(-5 + 72) = -132 + 735 - 603 = 0$$

Hence lines (1), (2) and (3) are concurrent.

Ex. 46 Prove using analytical geometry that the point of intersection of the diagonals of a trapezium lies on the line passing through the mid points of the parallel sides.

Sol. T.P.T. OC , AB & MN are concurrent

Equation of OC : $y = c/d\,x$ $\qquad cx - dy = 0$

(1)

Equation of AB : $y - 0 = \dfrac{2c}{2b - 2a}(x - 2a)$ $\qquad cx + (a - b)y - 2ac = 0$

(2)

Equation of MN : $y - 0 = \dfrac{2c}{b + d - a}(x - a)$ $\qquad 2cx + (a - b - d)y - 2ac = 0$

(3)

Now consider the determinant formed by the co-efficient of

$$\begin{vmatrix} c & -d & 0 \\ c & a-b & -2ac \\ 2c & a-b-d & -2ac \end{vmatrix} = -2ac^2 \begin{vmatrix} 1 & -d & 0 \\ 1 & a-b & 1 \\ 2 & a-b-d & 1 \end{vmatrix}$$

Using $R_3 \to R_3 - (R_1 + R_2)$ $\qquad -2ac^2 \begin{vmatrix} 1 & -d & 1 \\ 1 & a-b & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0$

Ex. 47 Prove that the three straight lines joining the angular points of a triangle to the middle points of the opposite sides meet in a point.

Sol. Let the angular points A, B, C be (x', y'), (x'', y''), (x''', y'''), respectively. Then, D, E, F the middle points of BC, CA, AB respectively, will be

$$\left(\frac{x'' + x'''}{2}, \frac{y'' + y'''}{2}\right), \left(\frac{x''' + x'}{2}, \frac{y''' + y'}{2}\right) \text{ and } \left(\frac{x' + x''}{2}, \frac{y' + y''}{2}\right).$$

The equation of AD will therefore be $\dfrac{y - y'}{\dfrac{y'' + y'''}{2} - y'} = \dfrac{x - x'}{\dfrac{x'' + x'''}{2} - x'}$

or $\quad y(x'' + x''' - 2x') - x(y'' + y''' - 2y') + x'(y'' + y''') - y'(x'' + x''') = 0.$

So the equations of BE and CF will be respectively

$y(x''' + x' - 2x'') - x(y''' + y' - 2y'') + x''(y''' + y') - y''(x''' + x') = 0.$

and $\quad y(x' + x'' - 2x''') - x(y' + y'' - 2y''') + x'''(y' + y'') - y'''(x' + x'') = 0.$

And, since the three equations when added together vanish identically, the three lines represented by them must meet in a point.

Ex. 48 Show that the area of the triangle whose sides are $a_ix + b_iy + c_i - 0$, i = 1, 2, 3 is equal to $\dfrac{\Delta^2}{2|C_1C_2C_3|}$, where C_1, C_2 and C_3 are the co-factors of C_1, C_2 and C_3 respectively in the determinant Δ, where $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

Sol. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta' = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$

where Δ' is the determinant of the co-factors of the elements of Δ.

The given lines are

$a_1x + b_1y + c_1 = 0$ ...(1)

$a_2x + b_2y + c_2 = 0$ ...(2)

$a_3x + b_3y + c_3 = 0$ ...(3)

Solving equations (1), (2) and (3) in pairs, we have the coordinates of the vertices of the triangle as

$$P \equiv \left(\frac{b_2c_3 - b_3c_2}{a_2b_3 - a_3b_2}, \frac{a_3c_2 - a_2c_3}{a_2b_3 - a_3b_2}\right) \equiv \left(\frac{A_1}{C_1}, \frac{B_1}{C_1}\right)$$

and similarly, we have

$$Q \equiv \left(\frac{A_2}{C_2}, \frac{B_2}{C_2}\right) \text{ and } R \equiv \left(\frac{A_3}{C_3}, \frac{B_3}{C_3}\right)$$

Now we have

$$\text{area of } \Delta\, PQR = \text{mod of } \frac{1}{2}\begin{vmatrix} \frac{A_1}{C_1} & \frac{B_1}{C_1} & 1 \\ \frac{A_2}{C_2} & \frac{B_2}{C_2} & 1 \\ \frac{A_3}{C_3} & \frac{B_3}{C_3} & 1 \end{vmatrix} = \frac{\Delta'}{2|C_1C_2C_3|} = \frac{\Delta^2}{2|C_1C_2C_3|} \quad [\because \Delta' = \Delta^2]$$

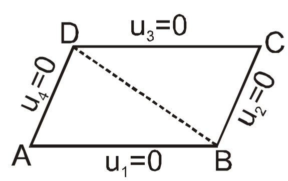
K. FAMILY OF LINES

The equation of a family of lines passing through the point of intersection of $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$ is given by $(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0$, where k is an arbitrary real number.

Note :

(i) If $u_1 = ax + by + c$, $u_2 = a'x + b'y + d$, $u_3 = ax + by + c'$, $u_4 = a'x + b'y + d'$ then, $u_1 = 0$; $u_2 = 0$; $u_3 = 0$; $u_4 = 0$ form a parallelogram.

$u_2\,u_3 - u_1\,u_4 = 0$ represents the diagonal BD.



On the similar lines $u_1u_2 - u_3u_4 = 0$ represents the diagonal AC.

(ii) The diagonal AC is also given by $u_1 + \lambda u_4 = 0$ and $u_2 + \mu u_3 = 0$, if the two equations are identical for some λ and μ.

[For getting the values of λ & μ compare the coefficients of x, y & the constant terms].

Ex. 49 If $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0$ then all lines represented by ax + by + c = 0 pass through a fixed point. Find the coordinates of that fixed point.

Sol. The determinant on solving gives a + c = 2b if a, b, c are in A.P.

now, $ax + by + 2b - a = 0$ or $a(x - 1) + b(y + 2) = 0$

hence the line passes through (1, – 2)

Ex. 50 Obtain the equations of the liens passing through the intersection of line 4x – 3y – 1 = 0 and 2x – 5y + 3 = 0 and equally inclined to the axes.

Sol. The equation of any line thorugh the intersection of the given lines is

$(4x - 3y - 1) + \lambda (2x - 5y + 3) = 0$

or $x(2\lambda + 4) - u(5\lambda + 3) + 3\lambda - 1 = 0$ ...(i)

Let m be the slope of this line. Then $m = \frac{2\lambda + 4}{5\lambda + 3}$

As the lne is equally inclined with the axes, therefore

$m = \tan 45°$ or $m = \tan 135°$ $\Rightarrow$ $m = \pm 1, \frac{2\lambda + 4}{5\lambda + 3} = \pm \lambda = -1$ or $\frac{1}{3}$, putting the values of λ in (i), we get $2x + 2y - 4 = 0$ and $14x - 14y = 0$

i.e. $x + y - 2 = 0$ and $x = y$ as the equations of the required lines. Ans.

Ex. 51 The equations of the two sides of a parallelogram are 3x – 10 y + 37 = 0 and 9x + 2 y – 17 = 0 and the equation of one of its diagonals is 3x – 2y – 19 = 0. Find the equations of two other sides of the parallelogram and the equation to its second diagonal.

Sol. equation of BD is $3x - 2y - 19 = 0$

AC will be perpendicular to BC and passing through (1, 4)

$\Rightarrow$ equation of AC = $2x + 3y - 14 = 0$

other two sides are $9x + 2y - 113 = 0$ &

$3x - 10y - 59 = 0$

Ex. 52 Two fixed lines OA (along x - axis) and OB as y = m x are cut by a variable line in the points P and Q respectively and M and N are the feet of the perpendiculars from P and Q upon the lines OQB and OPA. Show that if PQ passes through a fixed point (α, β) then MN will also pass through a fixed point.

Sol. $m_{PR} = m_{QR}$ $\frac{\beta}{\alpha - p} = \frac{\beta - mq}{\alpha - q}$; $-\beta q = -\alpha m q - p\beta + p m q$

$p\beta + (\alpha m - \beta) q = p m q$ — (1)

Equation of M P : $y - 0 = -\frac{1}{m}(x - p)$; $x + m y = p$

Solving with y = m x ; M $\left(\frac{p}{1+m^2}, \frac{mp}{1+m^2}\right)$

Equation of M N $y - 0 = \frac{\frac{mp}{1+m^2}}{\frac{p}{1+m^2} - q}(x - q)$

Simplifying $q(y + y m^2) + p(m x - y) = p q m$ — (2)

(1) – (2) gives $q(y(1 + m^2) - \alpha m + \beta) + p(m x - y - b) = 0$ i.e. $L_1 + k L_2 = 0$

L. BISECTORS OF THE ANGLES BETWEEN TWO LINES

(i) Equations of the bisectors of angles between the lines ax + by + c = 0 &

$a'x + b'y + c' = 0$ $(ab' \neq a'b)$ are : $\dfrac{ax + by + c}{\sqrt{a^2 + b^2}} = \pm \dfrac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$

(ii) **To discriminate between the acute angle bisector & the obtuse angle bisector**

If θ be the angle between one of the lines & one of the bisectors, find $\tan\theta$.

If $|\tan\theta| < 1$, then $2\theta < 90°$ so that this bisector is the acute angle bisector.

If $|\tan\theta| > 1$, then we get the bisector to be the obtuse angle bisector.

(iii) To discriminate between the bisector of the angle containing the origin & that of the angle not containing the origin. Rewrite the equations , ax + by + c = 0 & $a'x + b'y + c' = 0$ such that the constant terms c , c′ are positive. Then ;

$\dfrac{ax + by + c}{\sqrt{a^2 + b^2}} = + \dfrac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$ gives the equation of the bisector of the angle containing the origin & $\dfrac{ax + by + c}{\sqrt{a^2 + b^2}} = - \dfrac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$ gives the equation of the bisector of the angle not containing the origin.

(iv) To discriminate between acute angle bisector & obtuse angle bisector proceed as follows Write ax + by + c = 0 & $a'x + b'y + c' = 0$ such that constant terms are positive.

If $aa' + bb' < 0$, then the angle between the lines that contains the origin is acute and the equation of the bisector of this acute angle is $\dfrac{ax + by + c}{\sqrt{a^2 + b^2}} = + \dfrac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$

therefore $\dfrac{ax + by + c}{\sqrt{a^2 + b^2}} = - \dfrac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$ is the equation of other bisector.

If, however , $aa' + bb' > 0$, then the angle between the lines that contains the origin is obtuse & the equation of the bisector of this obtuse angle is :

$\dfrac{ax + by + c}{\sqrt{a^2 + b^2}} = + \dfrac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$; therefore $\dfrac{ax + by + c}{\sqrt{a^2 + b^2}} = - \dfrac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$

is the equation of other bisector.

(v) Another way of identifying an acute and obtuse angle bisector is as follows :

Let $L_1 = 0$ & $L_2 = 0$ are the given lines & $u_1 = 0$ and $u_2 = 0$ are the bisectors between $L_1 = 0$ & $L_2 = 0$. Take a point P on any one of the lines $L_1 = 0$ or $L_2 = 0$ and drop perpendicular on $u_1 = 0$ & $u_2 = 0$ as shown. If ,

$|p| < |q| \Rightarrow u_1$ is the acute angle bisector.

$|p| > |q| \Rightarrow u_1$ is the obtuse angle bisector.

$|p| = |q| \Rightarrow$ the lines L_1 & L_2 are perpendicular.

Note : Equation of straight lines passing through $P(x_1, y_1)$ & equally inclined with the lines $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$ are those which are parallel to the bisectors between these two lines & passing through the point P.

Ex. 53 Find the equations of the bisectors of the angle between the straight lines $3x - 4y + 7 = 0$ and $12x - 5y - 8 = 0$.

Sol. The equations of the bisectors of the angles between $3x - 4y + 7 = 0$ and $12x - 5y - 8 = 0$ are

$$\frac{3x-4y+7}{\sqrt{3^2+(-4)^2}} = \pm\frac{12x-5y-8}{\sqrt{12^2+(-5)^2}}$$

or $\dfrac{3x-4y+7}{5} = \pm\dfrac{12x-5y-8}{13}$ or $39x - 52y + 91 = \pm (60x - 25y - 8)$

Taking the positive sign, we get $21x + 27y - 131 = 0$ as one bisector Ans.

Taking the negative sign, we get $99x - 77y + 51 = 0$ as the other bisecotr. Ans.

Ex. 54 For the straight lines $4x + 3y - 6 = 0$ and $5x + 12y + 9 = 0$, find the equation of the

(i) bisector of the obtuse angle between them ;

(ii) bisector of the acute angle between them ;

Sol.(i) The equations of the given straight lines are

$4x + 3y - 6 = 0$ ...(1)

$5x + 12y + 9 = 0$ ...(2)

The equation of the bisectors of the angles between lines (1) and (2) are

$$\frac{4x+3y-6}{\sqrt{4^2+3^2}} = \pm\frac{5x+12y+9}{\sqrt{5^2+12^2}} \text{ or } \frac{4x+3y-6}{5} = \pm\frac{5x+12y+9}{13}$$

Taking the positive sign, we have $\dfrac{4x+3y-6}{5} = \dfrac{5x+12y+9}{13}$

or $52x + 39y - 78 = 25x + 60y + 45$ or $27x - 21y - 123 = 0$

or $9x - 7y - 41 = 0$

Taking tyhe negative sign, we have $\dfrac{4x+3y-6}{5} = -\dfrac{5x+12y+9}{13}$

or $52x + 39y - 78 = -25x - 60y - 45$ or $77x + 99y - 33 = 0$

or $7x + 9y - 3 = 0$

Hence the equation of the bisectors are

$9x - 7y - 41 = 0$ ...(3)

and $7x + 9y - 3 = 0$ ...(4)

Now slope of line (1) $= -\dfrac{4}{3}$ and slope of the bisector (3) $= \dfrac{9}{7}$.

If θ be the acute angle between the line (1) and the bisector (3), then

$$\tan\theta = \left|\frac{\frac{9}{7}+\frac{4}{3}}{1+\frac{9}{7}\left(-\frac{4}{3}\right)}\right| = \left|\frac{27+28}{21+36}\right| = \left|\frac{55}{-15}\right| = \frac{11}{3} > 1$$

$\therefore$ $\theta > 45°$

Hence $9x - 7y - 41 = 0$ is the bisector of the obtuse angle between the given line (1) and (2) Ans.

(ii) Since $9x - 7y - 41 = 0$ is the bisector of the obtuse angle between the given lines, therefore the other bisecotrs $7x + 9y - 3 = 0$ will be the bisector of the acute angle between the given lines.

2nd Method

Writing the equation of the lines so that constants become positive we have

$-4x - 3y + 6 = 0$ ...(1)

and $5x + 12y + 9 = 0$ ...(2)

Here $a_1 = -4$, $a_2 = 5$, $b_1 = -3$, $b_2 = 12$

Now $a_1a_2 + b_1b_2 = -20 - 36 = -56 < 0$

$\therefore$ origin does not lie in the obtuse angle between lines (1) and (2) and hence equation of the bisector of the obtuse angle between lines (1) and (2) will be

$$\frac{-4x-3y+6}{\sqrt{(-4)^2+(-3)^2}} = -\frac{5x+12y+9}{\sqrt{5^2+12^2}}$$

or $13(-4x-3y+6) = -5(5x+12y+9)$

or $27x - 21y - 123 = 0$ or $9x - 7y - 41 = 0$ Ans.

and the equation of the bisector of the acute angle will be (origin lies in the acute angle)

$$\frac{-4x-3y+6}{\sqrt{(-4)^2+(-3)^2}} = -\frac{5x+12y+9}{\sqrt{5^2+12^2}}$$

or $77x + 99y - 33 = 0$ or $7x + 9y - 3 = 0$ Ans.

Ex. 55 For the straightlines $4x + 3y - 6 = 0$ and $5x + 12y + 9 = 0$, find the equation of the bisector of the angle which contains the origing.

Sol. For point O(0,0), $4x + 3y - 6 = -6 < 0$ and $4x + 12y + 9 = 9 > 0$

Hence for point O(0, 0) $4x + 3y - 6$ and $5x + 12y =+ 9$ are of opposite signs.

Hence equation of the bisector of the angles between the given lines containing the origin will be

$$\frac{4x+3y-6}{\sqrt{(4)^2+(3)^2}} = -\frac{5x+12y+9}{\sqrt{5^2+12^2}}$$

or $\dfrac{4x+3y-6}{5} = -\dfrac{5x+12y+9}{13}$ or $52x + 39y - 78 = -25x - 60y - 45$

or $77x + 99y - 33 = 0$ or $7x + 9y - 3 = 0$ Ans.

Ex. 56 Find the equations to the straight lines passing through the foot of the perpendicular from the point (h, k) upon the straight line $Ax + By + C = 0$, and bisecting the angles between the perpendicular and the given straight line.

Sol. Equation of the given line is $Ax + By + C = 0$. ...(1)

Equation of any line perpendicular to (1) will be $Bx - Ay = \lambda$, where λ is any constant. As the perpendicular line passes through (h, k), hence it will satisfy (2). So $Bh - Ak = \lambda$. Substituting in (2), we get the equation of the line perpendicular to (1) and passing through (h, k) as

$Bx - Ay = Bh - Ak$

or $Bx - Ay - Bh + Ak = 0$. ...(3)

Equations of the bisectors of the angles between the lines given by (1) and (3) will be

$$\frac{Ax+By+C}{\sqrt{(A^2+B^2)}} = \pm\left[\frac{Bx-Ay-Bh+Ak}{\sqrt{(B^2+A^2)}}\right]$$

or $Ax + By + C = \pm [B(x - h) - A(y - k)]$

or $A(y - k) - B(x - h) = \pm (Ax + By + C)$.

Ex. 57 Find the centre of the circle inscribed in the triangle whose angular points A, B, C are respectively the points (1, 2), (25, 8) and (9, 21).

The equations of the sides BC, CA, AB will be found to be

$13x + 16y - 453 = 0$, $19x - 8y - 3 = 0$ and $x - 4y + 7 = 0$.

Sol. If the co-ordinates of A, B, C be subsituted in the left-hand members of these equations, the results will be –, +, – respectively.

Now change the signs of all the terms in the equations fo the lines, if necessary, so that each vertex will be on the positive side of the opposite line; the equations will then be

$-12x - 16y + 453 = 0$, $19x - 8y - 3 = 0$ and $-x + 4y - 7 = 0$.

Then $$\frac{-13x-16y+453}{\sqrt{(13^2+16^2)}} = +\frac{19x-8y-3}{\sqrt{(19^2+8^2)}}$$

must be the internal bisector of the angle ACB, for both members of the equation must be positive or both must be negative, so that any point on the bisector must be on the positive side of both CA and CB, or on the negative side of both.

So also $$\frac{19x-8y-3}{\sqrt{(19^2+8^2)}} = +\frac{-x+4y-7}{\sqrt{(1^2+4^2)}}$$

is te internal bisector of the angle BAC.

Hence the centre of the inscribed circle is given by

$$\frac{-13x-16y+453}{5\sqrt{17}} = \frac{19x-8y-3}{5\sqrt{17}} = \frac{-x+4y-7}{\sqrt{17}},$$

and the point will be found to be (11.5, 11).

Ex. 58 Let ABC be a triangle such that the coordinates of the vertex A are (–3, 1). Equation of the median through B is 2x + y – 2 = 0 and equation of the angular bsector of C is 7x – 4y – 1 = 0. Find the equation of the sides of the triangle.

Sol. Since C lies on 7x – 4y – 1 = 0, therefore let us choose its coordinates as $\left(h, \frac{7h-1}{4}\right)$.

The mid-point of AC, i.e. $\left(\frac{h-3}{2}, \frac{7h+3}{8}\right)$ lies on 2x + y – 3 = 0, therefore

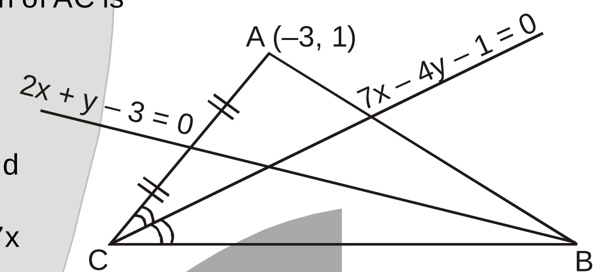
we have $$2\left(\frac{h-3}{2}\right) + \left(\frac{7h+3}{8}\right) - 3 = 0$$

gives h = 3.

Hence, coordinates of C are (3, 5) and equation of AC is

$$y - 5 = \frac{5-1}{3+3}(x-3)$$

i.e. $\quad 2x - 3y + 9 = 0 \qquad \text{...(1)}$



Let slope of BC = m. Since lines BC and AC $\left(\text{slope} = \frac{2}{3}\right)$ are equally inclined to the line 7x – 4y – 1 = 0 $\left(\text{slope} = \frac{7}{4}\right)$, therefore we have

i.e. $$\frac{m-\frac{7}{4}}{1+\frac{7m}{4}} = \frac{\frac{7}{4}-\frac{2}{3}}{1+\frac{7}{6}} \text{ (see fig.)} \qquad \text{i.e.} \qquad \frac{4m-7}{7m+4} = \frac{1}{2} \text{ gives } m = 18.$$

Therefore, equation of BC is

$$y - 5 = 18(x-3)$$

i.e. $\quad 18x - y - 49 = 0 \qquad \text{...(2)}$

Solving equations (2) and 2x + y – 3 = 0 simultaneously gives the coordinates of B as $\left(\frac{13}{5}, \frac{-11}{5}\right)$.

Therefore, equation of AB is $y - 1 = \left(\frac{1+\frac{11}{5}}{-3-\frac{13}{5}}\right)(x+3)$

i.e. $\quad y - 1 = \frac{-4}{7}(x+3) \qquad$ i.e. $\quad 4x + 7y + 5 = 0. \qquad \text{...(3)}$

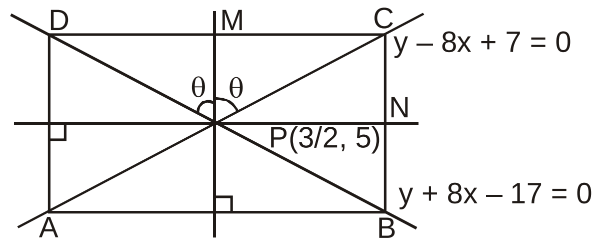
Ex. 59 The equation of the diagonals a rectangle are $y + 8x - 17 = 0$ and $y - 8x + 7 = 0$
If the area of the rectangle is 8 sq. units, find the equation of the sides of the rectangle.

Sol. The intersection point of the given diagonals

$$y + 8x - 17 = 0$$

and $$y - 8x + 7 = 0$$

is $$P \equiv \left(\frac{3}{2}, 5\right).$$



Now, we have $\Delta CPD = \frac{1}{4}$ (ABCD) = 2 sq. units

i.e. $l^2 \sin\theta \cos\theta = 2$ [putting CP = l]

i.e. $l^2 \sin 2\theta = 4$ ...(1)

where $\tan 2\theta = \left|\frac{8-(-8)}{1+(8)(-8)}\right| = \frac{16}{63}$ therefore $\sin 2\theta = \frac{16}{65}$ and $\cos 2\theta = \frac{63}{65}$

Putting in equation (1), we have $l^2 = \frac{65 \times 4}{16} = \frac{65}{4}$ i.e. $l = \frac{\sqrt{65}}{2}$.

Therefore, we have

$$PM = l\cos\theta = l\sqrt{\frac{1+\cos 2\theta}{2}} = \frac{\sqrt{65}}{2} \times \sqrt{\frac{65+63}{2\times 65}} = 4$$

and $$PN = l\sin\theta = l\sqrt{\frac{1-\cos 2\theta}{2}} = \frac{\sqrt{65}}{2} \times \sqrt{\frac{65-63}{2\times 65}} = \frac{1}{2}$$

Equation of the angular bisectors of the diagonals are

$$\frac{y+8x-17}{\sqrt{65}} = \pm\frac{y-8x+17}{\sqrt{65}} \quad \text{i.e.} \quad x = \frac{3}{2} \text{ and } y = 5.$$

From the fig we can see that sides AB, CD are lines parallel to the angular bisector PN $\equiv y - 5 = 0$ at a distance of PN = 4 units. Hence, their equations are

$$y = 5 \pm 4 \quad \text{i.e.} \quad y = 1,\ y = 9$$

and the sides AD, BC are lines parallel to the angular bisector PM $\equiv x - 3/2 = 0$ at a distace of PN = 1/2 units. Hence, their equations are

$$x = \frac{3}{2} \pm \frac{1}{2} \quad \text{i.e.} \quad x = 1,\ x = 2.$$

M. Locus

The locus of a moving point is the path traced out by it under certain geometrical condition or conditions.

If a point moves in a plane under the geometrical condition that its distance from a fixed point O in the plane is always equal to a constant quantity a, then the curve traced out by the moving point will be circle with centre O and radius a. Thus locus of the point is a circle with centre O and radius a.

Equation of a locus : An equation is said to be the equation of the locus of a moving point if the following two conditions are satisfied

(i) The co-ordinates of every point on the locus satisfy the equation.

(ii) If the co-ordinates of any point satisfy the equation, then that point must lie on the locus.

Working Rule :

(i) If x and y co-ordinates of the moving point are given in terms of a third variable t (called the parameter), eliminate t to obtain the relation in x and y and simplify this relation. This will give the required locus.

(ii) If some geometrical conditions are given and we have to find the locus, then

(a) Take the co-ordinates of the variable point as (α, β).

(b) Write down the given geometrical conditions and express these conditions in terms of α and β.

(c) Eliminate the variable to get the relation in α and β i.e. this relation must contain only α, β and known quantities.

(d) Finally put x in place of α and y in place of β and the equation thus obtained will be the required equation of the locus.

(e) Sometimes the co-ordinates of the moving point itself is taken as (x, y). But this should be done only when no equation is given in question and co-ordinate of no point is given as (x, y). In this case the relation in x and y can be directly obtained by eliminating the variable.

(iii) Make suitable choice of the origin and the axes if co-ordinates of no point and equation of no curve is given in the question.

Ex. 60 A and B being the fixed points (a, 0) and (–a, 0) respectively, obtain the equation giving the locus of P, when PA = nPB, n being constant.

Sol. The given relation is PA = n PB or $PA^2 = n^2 PB^2$

or $[x - a)^2 + (y - 0)^2] = n^2 [(x + a)^2 + (y - 0)^2]$ (on putting the values of PA and PB)

or $x^2 + a^2 - 2ax + y^2 = n^2 [x^2 + a^2 + 2ax + y^2]$

or $0 = x^2 (n^2 - 1) + y^2 (n^2 - 1) + a^2 (n^2 - 1) + 2ax (n^2 + 1)$

or $(n^2 - 1) (x^2 + y^2 + a^2) + 2ax (n^2 + 1) = 0$

Ex. 61 The base BC (=2a) of a triangle ABC is fixed; the axes being BC and a perpendiculat to it through its middle point, find the locus of the vertex A, when the difference of the base angles is given (= α).

Sol. Let the co-ordinates of A be (x, y) and $\angle ABC = \theta_1$ and $\angle ACB = \theta_2$. Then

$$\tan \theta_1 = \frac{y-0}{x+a} \text{ and } \tan (180 - \theta_2) = \frac{y-0}{x-a}$$

$$\therefore \quad \tan \theta_1 = \frac{y}{x+a} \text{ and } \tan \theta_2 = \frac{-y}{x-a}.$$

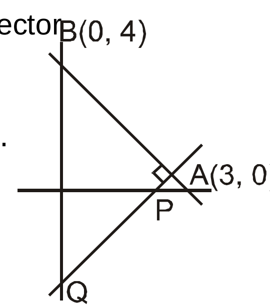
By hypothesis, $\theta_2 - \theta_1 = \alpha$ or $\tan (\theta_2 - \theta_1) = \tan \alpha$

or $$\frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1} = \tan \alpha$$ or $$\frac{-\frac{y}{x-a} - \frac{y}{x+a}}{1 - \frac{y}{x-a}\frac{y}{x+a}} = \tan \alpha$$

or $$\frac{-2xy}{x^2 - a^2 - y^2} = \frac{1}{\cos\alpha}$$ or $x^2 + 2xy\cot \alpha - y^2 = a^2$.

Ex. 62 The line $L_1 \equiv 4x + 3y - 12 = 0$ intersects the x and the y-axis at A and B respectively. A variable line perpendicular to L_1 intersects the x and the y-axis at P and Q respectively. Find the locus of the circumcentre of triangle ABQ.

Sol. Clearly circumcentre of triangle ABQ will lie on the perpendicular bisector of line AB.

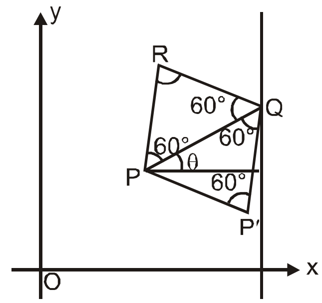


Now equation of perpendicular bisector of line AB is $3x - 4y + \frac{7}{2} = 0$.

Hence locus of circumcentre is $6x - 8y + 7 = 0$.

Ex. 63 An equilateral triangle PQR is formed where P(1, 3) is fixed point and Q is moving point on line x = 3. Find the locus of R.

Sol. Slope of PR is $(\theta \pm 60°)$.



Let coordinates of R be (h, k).

$$\frac{h-1}{\cos(\theta \pm 60°)} = \frac{k-2}{\sin(\theta \pm 60°)} = 2\sec\theta$$

h = 1 + 2 cos $(\theta \pm 60^o)$ sec θ

$\Rightarrow \quad h = 1 + 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\tan\theta\right)$ ...(1)

$\Rightarrow \quad k = 2 + 2\sin(\theta \pm 60^o)\sec\theta$

$\Rightarrow \quad k = 2 + 2\left(\frac{\tan\theta}{2} \mp \frac{\sqrt{3}}{2}\right)$ ....(2)

Eliminating θ from (1) and (2), $\quad \frac{h-2}{\sqrt{3}} = \pm\tan\theta$

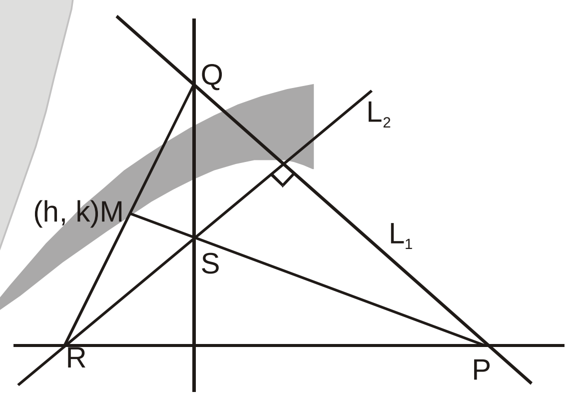
$k - 2 \pm \sqrt{3} = \tan\theta \quad \Rightarrow \quad \frac{h-2}{\sqrt{3}} = \pm(k - 2 \pm \sqrt{3})$

Locus is $(x-2) = \pm\sqrt{3}\,(y - 2 \pm \sqrt{3})$.

Ex. 64 Let a given line L_1 intersect the X–axis at P and Q respectively. Let a variable line L_2 perpendicular to L_1 cut the X–axis and the Y–axis at R and S respectively. Find the locus of the intersection point of the lines PS and QR.

Sol. Let the equation of the given

line L_1 be $\frac{x}{a} + \frac{y}{b} = 1$ ....(1)



The intersection points of this line with the X-axis and the Y-axis are P(a, 0) and Q(0, b) respectively.

Let the equation of the line L_2 perpendicular to L_1 be $\frac{x}{a} - \frac{y}{b} = \lambda$ (λ is a variable) ....(2)

The intersection point of L_2 with the X–axis and the Y–axis are R $(\lambda b, 0)$ and S$(0, -\lambda a)$ respectively.

Let M (h, k) be the coordinates of the point whose locus is to be found.

Since M lies on PS, therefore we have

i.e. $\quad \lambda = \frac{k}{h-a}$ ...(3)

Also, since M lies on QR, therefore we have

slope of MQ = slope of QR

i.e. $\quad \frac{k-b}{h} = \frac{1}{\lambda}$ ...(4)

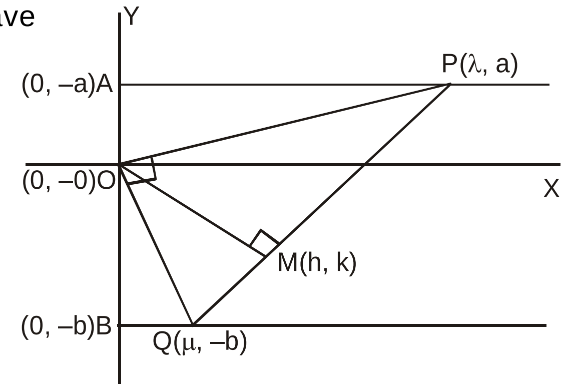
Multiplying equations (3), (4) and then putting (x, y) in place of (h, k) gives the equation of the required locus as x(x – a) + y(y – b) = 0.

Ex. 65 A, O and B are fixed points in a straight line. A point P is chosen on a line passing through A perpendicular to AOB, and a point Q is chosen on a line pasing through B perpendicular to AOB such that $\angle$POQ is a right angle. Find the locus of the foot of the perpendicular from O on PQ.

Sol. Let us choose the fixed point O as the origin and line AOB as the Y–axis. Also, let OA = a and OB = b (see fig.).

Let us choose the variable points as P(λ, a) and Q(μ, –b).

Since OP is perpendicular to OQ, therefore we have



i.e. $\left(\dfrac{-b}{\mu}\right)\left(\dfrac{a}{\lambda}\right) = -1$

i.e. $\mu\lambda = ab$ ...(1)

Let M(h, k) be the point whose locus is to be found.

Since M lies on AB, therefore we have

slope of MQ = slope of PQ

i.e. $\dfrac{(k+b)}{(h-\mu)} = \dfrac{(a+b)}{(\lambda-\mu)}$ i.e. $\lambda(k + b) + \mu(a - k) = (a + b)h$...(2)

Also, since OM is perpendicular to PQ, therefore we have

$\dfrac{(a+b)}{(\lambda-\mu)} = -\dfrac{h}{k}$ i.e. $\mu - \lambda = \dfrac{k}{h}(a + b)$ ...(3)

Solving equations (2), (3) simultaneously, gives $\lambda = \dfrac{(h^2+k^2-ak)}{h}$ and $\mu = \dfrac{(h^2+k^2-bk)}{h}$

Putting the above values of λ, μ in equation (1), we have

$(h^2 + k^2 - ak)(h^2 + k^2 + bk) = (h^2)\,ab$

i.e. $(h^2 + k^2)\{h^2 + k^2 + k(b - a) - ab\} = 0$

i.e. $h^2 + k^2 + k(b - a) - ab = 0$

Putting (x, y) in place of (h, k) gives the equation of the required locus as

$x^2 + y^2 + y(b - a) - ab = 0.$

Ex. 66 Through a fixed point any straight line is drawn meeting two given parallel straight lines in P and Q; through P and Q straight lines are drawn in fixed directions, meeting in R: prove that the locus of R is a straight line.

Sol. Take the fixed point O for origin, and the axis of y parallel to the two parallel straight lines; and let the equations of these parallel lines be x = a, x = b.

Then, if the equation of OPQ be y = mx, the abscissa of Q is b, and therefore its ordinate mb.

Let PR be always parallel to y = m′x and QR always parallel to y = m″x, then the equation of PR will be $y - ma = m'(x - a)$ ....(i),

$y - mb = m''(x - b)$ ....(ii).

To find locus we have only to eliminate m between the equation (i) and (ii).

The result is $(b - a)\,y = m'b\,(x - a) - m''a\,(x - b)$.

This equation is of the first degree, and therefore the required locus is a straight line.

Ex. 67 Find the equation of the straight line through the point A (1 , 0) and perpendicular to the line PQ given by the equation , $y = mx + m^3$. Obtain the co-ordinates of the point R in which the lines meet and prove that the locus of R as m varies is $y^2 = 1 - x$. The line through R parallel to the x – axis meets the line through the origin parallel to PQ in S. Find the co-ordinates of S and prove that when m is positive, the radius of the circle circumscribing the Δ ASR is $\dfrac{m}{2}\sqrt{1+m^2}$.

Sol. $y = -\frac{1}{m}(x - 1)$ or $x + my - 1 = 0$

Solving $y = mx + m^3$ — (1) and

$x + my = 1$ — (2)

co-ordinates of R $(1 - m^2, m)$

$\therefore h = 1 - m^2$ & $k = m$ $\Rightarrow$ Locus in $y^2 = 1 - x$

Note that ΔASR is a right triangle

$$\Rightarrow r = \frac{1}{2} AR = \frac{1}{2}\sqrt{m^2 + m^4} = \frac{1}{2} m\sqrt{1 + m^2}$$

$my + x = 1$; R $(1 - m^2, m)$; S $(1, m)$

Ex. 68 A straight line is drawn parallel to the base of a given triangle and its extremities are joined transversely to those of the base. Show that the locus of the point of intersection of the joining lines is a straight line.

Sol. Equation of BE :

$y = \frac{\lambda b}{\lambda a + c} x$ or $k(\lambda a + c) = h\lambda b$ $\Rightarrow \lambda = \frac{kc}{bh - ka}$

Similarly equation of CD :

$y - 0 = \dfrac{\frac{\lambda b}{1+\lambda} - 0}{\frac{\lambda a}{1+\lambda} - c}(x - c)$ $k\left(\frac{\lambda a}{1+\lambda} - c\right) = \frac{\lambda b}{1+\lambda}(h - c)$ $\Rightarrow$ $\lambda = \frac{kc}{(a-c)k + bc - bh}$

Equating the two values of λ we get the locus of P (h, k) as

$2bx - (2a - c)y = bc$. which is a straight line

Ex. 69 'Q' is any point on the line $x = a$. If A is the point $(a, 0)$ and QR, the bisector of the angle OQA meets the axis of x in R, then show that the locus of the foot of the perpendicular from R to OQ has the equation, $(x - 2a)(x^2 + y^2) + ax^2 = 0$.

Sol. Equation of OQ, $y = \frac{k}{h} x$;

It passes through $(a, \lambda) \Rightarrow \lambda = \frac{ka}{h}$

Note that Δ's $\cong \Delta$ QRA

Hence QP = QA $\Rightarrow \frac{a^2k^2}{h^2} = (h - a)^2 + \left(k - \frac{ak}{h}\right)^2$. Simplify to get locus

Ex. 70 Through a fixed point O are drawn two straight lines at right angles to meet two fixed straight lines, which are also at right angles, in the points P and Q. Show that the locus of the foot of the perpendicular from O on PQ is a straight line.

Sol. Let us take the line passing through O be the axes, O′P and O′Q be the other pair, O′ being (a, b) and cutting the axes at P and Q.

Let the equation of O′Q be, $(y - b) = m(x - a)$. ...(1)

Hence the equation of O′P will be $(y - b) = -\frac{1}{m}(x - a)$. ...(2)

(as PO′ is perpendicular to QO′)

Solving (1) with y-axis, OQ willbe $(b - am)$. Similarly solving (2) with x-axis, OP will be $(a + bm)$.

Hence equation of OP will be $\frac{x}{a+bm}+\frac{y}{b-am}=1$. ...(3)

Let R be the perpendicular on OP from O and its co-ordinates be (h, k). As R lies on OP, (h, k) will satisfy its equation. Hence $\frac{h}{a+bm}+\frac{k}{b-am}=1$. ...(4)

Slope of OR = $\frac{k}{h}$. Slope of QP = $\frac{b-am}{a+bm}$.

As OR is perpendicualr to QR, $\left(\frac{k}{h}\right)\times\left(-\frac{b-am}{a+bm}\right)=-1$. ...(5)

Eliminating m from (4) and (5), we get the required locus.

N. A Pair Of Straight Lines Through Origin

(i) A homogeneous equation of degree two of the type $ax^2 + 2hxy + by^2 = 0$ always represents a pair of straight lines passing through the origin & if :

(a) $h^2 > ab \Rightarrow$ lines are real & distinct.

(b) $h^2 = ab \Rightarrow$ lines are coincident.

(c) $h^2 < ab \Rightarrow$ lines are imaginary with real point of intersection i.e. (0, 0)

(ii) If $y = m_1x$ & $y = m_2x$ be the two equations represented by $ax^2 + 2hxy + by^2 = 0$, then ;

$m_1 + m_2 = -\frac{2h}{b}$ & $m_1 m_2 = \frac{a}{b}$.

(iii) If θ is the acute angle between the pair of straight lines represented by ,

$ax^2 + 2hxy + by^2 = 0$, then ; $\tan\theta = \left|\frac{2\sqrt{h^2-ab}}{a+b}\right|$.

The condition that these lines are :

(a) At right angles to each other is $a + b = 0$. i.e. co–efficient of x^2 + co–efficient of $y^2 = 0$.

(b) Coincident is $h^2 = ab$.

(c) Equally inclined to the axis of x is $h = 0$. i.e. coeff. of $xy = 0$.

(iv) The equation to the straight lines bisecting the angle between the straight lines ,

$ax^2 + 2hxy + by^2 = 0$ is $\frac{x^2-y^2}{a-b}=\frac{xy}{h}$.

Proof : Let $ax^2 + 2hxy + by^2 \equiv a(x-\alpha y)(x-\beta y)$,

$\therefore \quad \alpha+\beta=-\frac{2h}{a}$ and $\alpha\beta=\frac{b}{a}$.

The equation of the bisectors is, $\frac{(x-\alpha y)^2}{1+\alpha^2}-\frac{(x-\beta y)^2}{1+\beta^2}=0$.

That is $(1+\beta^2)(x^2-2\alpha xy+\alpha^2y^2)-(1+\alpha^2)(x^2-2\beta xy+\beta^2y^2)=0$,

that is $x^2+2\frac{1+\alpha\beta}{\beta+\alpha}xy-y^2=0$,

that is $x^2+2\frac{1-\frac{b}{a}}{-\frac{2h}{a}}xy-y^2=0$,

that is $\frac{x^2-y^2}{a-b}=\frac{xy}{h}$.

(v) The product of the perpendiculars, dropped from (x_1, y_1) to the pair of lines represented by the equation, $ax^2 + 2hxy + by^2 = 0$ is $\dfrac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a-b)^2 + 4h^2}}$.

Note : A homogeneous equation of degree n represents n straight lines passing through origin.

O. GENERAL EQUATION OF SECOND DEGREE REPRESENTING A PAIR OF STRAIGHT LINES

(i) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if :

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$, i.e. if $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$..

(ii) The angle θ between the two lines representing by a general equation is the same as that between the two lines represented by its homogeneous part only.

Ex. 71 Prove that the following equation represents two straight lines; find also their point of intersection and the angle between them : $6y^2 - xy - x^2 + 30y + 36 = 0$.

Sol. $6y^2 - xy - x^2 + 30y + 36 = 0$.

Hence a = –1, b = 6, c = 36, h = –1/2, g = 0, g = 15,

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= (-1).6.36 + 2.15.0\left(-\dfrac{1}{2}\right) -)-1)\ (15)^2 - 6.(0)^2 - 36\left(-\dfrac{1}{2}\right)^2 = 0$

Hence the equation represents two straight lines.

Again $6y^2 - xy - x^2 = (3y + x)(2y - x)$

Hence let $6y^2 - xy - x^2 + 30y + 36 \equiv (3y + x + A) \times (2y - x + B)$. ...(1)

Comparing coefficients of x and y, we get $B - A = 0$ ...(2)

and $3 + 2A = 30$ ...(3)

Solving, we get A = 6 and B = 6.

Substituting in (1), we get the equations of the line represented as

$3y + x + 6 = 0$ ...(4)

and $2y - x + 6 = 0$. ...(5)

If θ be the angle betwee then $\tan\theta = \dfrac{2\sqrt{\left[\left(-\frac{1}{2}\right)^2 - (-1)(6)\right]}}{-1+6} = \dfrac{2\sqrt{\left(\frac{25}{4}\right)}}{5} = 1$.

$\therefore$ $\theta = 45°$. Ans.

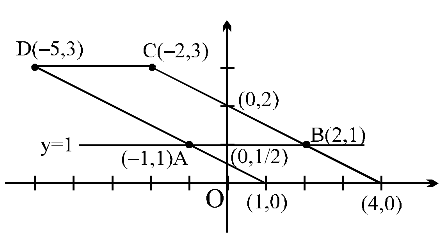
Solving (4) and (5), we get $x = \dfrac{6}{5}$, $y = -\dfrac{12}{5}$. So point of intersection is $\left(\dfrac{6}{5}, -\dfrac{12}{5}\right)$.

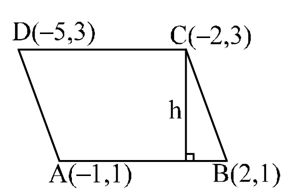
Ex. 72 The two line pairs $y^2 - 4y + 3 = 0$ and $x^2 + 4xy + 4y^2 - 5x - 10y + 4 = 0$ enclose a 4 sided convex polygon find (i) area of the polygon (ii) length of its diagonals.

Sol. $y^2 - 4y + 3 = 0$ and $x^2 + 4xy + 4y^2 - 5x - 10y + 4 = 0$

$(y - 3)(y - 1)$ $(x + 2y - 1)(x + 2y - 4) = 0$

$y = 1, y = 3$





$l(AB) = 3$ $\qquad h = 2$

$\therefore$ area of parallelogram = 3 × 2 = 6 Ans.

Length of AC = $\sqrt{1^2 + 2^2} = \sqrt{5}$ $\qquad$ length of BD = $\sqrt{7^2 + 2^2} = \sqrt{53} = \sqrt{53}$

Ex. 73 Show that the equation $6x^2 - 5xy + y^2 = 0$ represents a pair of distinct straight lines, each passing through the origin. Find the separate equations of these lines.

Sol. The given equation is a homogeneous equation of second degree. So, it represents a pair of straight lines passing through theorigin. Comparing the given equation with $ax^2 + 2hxy + by^2 = 0$, we obtain a = 6, b = 1 and 2h = –5.

$$\therefore \quad h^2 - ab = \frac{25}{4} - 6 = \frac{1}{4} > 0 \quad \Rightarrow \quad h^2 > ab$$

Hence, the given equation represents a pair of distinct lines passing through the origin.

Now, $6x^2 - 5xy + y^2 = 0 \quad \Rightarrow \quad \left(\frac{y}{x}\right)^2 - 5\left(\frac{y}{x}\right) + 6 = 0$

$$\Rightarrow \quad \left(\frac{y}{x}\right)^2 - 3\left(\frac{y}{x}\right) - 2\left(\frac{y}{x}\right) + 6 = 0 \quad \Rightarrow \quad \left(\frac{y}{x} - 3\right)\left(\frac{y}{x} - 2\right) = 0$$

$$\Rightarrow \quad \frac{y}{x} - 3 = 0 \text{ or } \frac{y}{x} - 2 = 0 \Rightarrow y - 3x = 0 \text{ or } y - 2x = 0$$

So the given equation represents the straight lines y – 3x = 0 and y – 2x = 0 Ans.

Ex. 74 Find the equations to the pair of lines through the origin which are perpendicular to the lines represented by $2x^2 - 7xy + 2y^2 = 0$.

Sol. We have $2x^2 - 7xy + 2y^2 = 0$

$\Rightarrow \quad 2x^2 - 6xy - xy + 3y^2 = 0 \quad \Rightarrow \quad 2x(x - 3y) - y(x - 3y) = 0$

$\Rightarrow \quad (x - 3y)(2x - y) = 0 \quad \Rightarrow \quad x - 3y = 0 \text{ or } 2x - y = 0$

Thus the given equation represents the lines x – 3y = 0 and 2x – y = 0. The equations of the lines passing through the origin and perpendicular to the given lines are y – 0 = –3 (x – 0)

and $y - 0 - \frac{1}{2}(x - 0)$ [$\because$ (Slope of x – 3 y = 0) is 1/3 and (Slope of 2x – y = 0) is 2]

$\Rightarrow \quad y + 3x = 0$ and $2y + x = 0$ Ans.

Ex. 75 Find the angle between the pair of straight lines $4x^2 + 24xy + 11y^2 = 0$

Sol. Given equation is $4x^2 + 24xy + 11y^2 = 0$

Here a = coeff. of x^2 = 4, b = coeff. of y^2 = 11

and 2h = coeff. of xy = 24 $\quad \therefore \quad$ h = 12

$$\text{Now } \tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right| = \left|\frac{2\sqrt{144 - 44}}{4 + 11}\right| = \frac{4}{3}$$

Where θ is the acute angle between the lines.

$\therefore$ acute angle between the lines is $\tan^{-1}\left(\frac{4}{3}\right)$ and obtuse angle between them is

$\pi - \tan^{-1}\left(\frac{4}{3}\right)$ Ans.

Ex. 76 Find the equation of the bisectors of the angle between the lines represented by $3x^2 - 5xy + y^2 = 0$

Sol. Given equation is $3x^2 - 5xy + y^2 = 0$ ...(1)

comparing it with the equation $ax^2 + 2hxy + by^2 = 0$ ...(2)

we have a = 3, 2h = –5; and b = 4

Now the equation of the bisectors of the angle between the pair of liens (1) is $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$

or $\frac{x^2 - y^2}{3 - 4} = \frac{xy}{-\frac{5}{2}}$; or $\frac{x^2 - y^2}{-1} = \frac{2xy}{-5}$

or $5x^2 - 2xy - 5y^2 = 0$ Ans.

Ex. 77 Prove that the equation $2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$ represents a pair of straight lines. Find the co-ordinates of their point of intersection and also the angle between them.

Sol. Given equation is $2x^2 + 5xy + 2y^2 + 6x + 7y + 4 = 0$

Writing the equation (1) as a quadratic equation in x we have

$$2x^2 + (5y + 6)x + 3y^2 + 7y + 4 = 0$$

$$\therefore \quad x = \frac{-(5y+6) \pm \sqrt{(5y+6)^2 - 4.2(3y^2 + 7y + 4)}}{4}$$

$$= \frac{-(5y+6) \pm \sqrt{25y^2 + 60y + 36 - 24y^2 - 56y - 32}}{4}$$

$$= \frac{-(5y+6) \pm \sqrt{y^2 + 4y + 4}}{4} = \frac{-(5y+6) \pm (y+2)}{4}$$

$$\therefore \quad x = \frac{-5y - 6 + y + 2}{4}, \frac{-5y - 6 - y - 2}{4}$$

or $4x + 4y + 4 = 0$ and $4x + 6y + 8 = 0$

or $x + y + 1 = 0$ and $2x + 3y + 4 = 0$

Hence equation (1) represents a pair of straight lines whose equation are $x + y + 1 = 0$ ....(1)

and $2x + 3y + 4 = 0$ ....(2) Ans.

Solving these two equations, the required point of intersection is (1, –2) Ans.

Ex. 78 Find the equation of the pair of lines both of which pass through the point (1, –1) and are parallel to the angular bisectors of the line given by the equation

$$x^2 - y^2 + 4x - 2y + 3 = 0.$$

Sol. The given equation can be written as

$$x^2 + 4x + y^2 + 2y = 3$$

i.e. $(x + 2)^2 = y^2 + 2y - 3 + 4 = (y + 1)^2$

i.e. $(x + 2) = \pm(y + 1)$

gives the equations of the two lines as $x - y + 1 = 0$

and $x + y + 3 = 0$

Equation of the angular bisectors of the above lines are given by

$$\frac{x - y + 1}{\sqrt{2}} = \pm \frac{x + y + 3}{\sqrt{2}}$$

i.e. $x + 2 = 0$ and $y + 1 = 0$.

Equations of th lines passing through the point (1, –1) and parallel to the angular bisectors are

$$x - 1 = 0 \text{ and } y + 1 = 0$$

Jointly, the required equation is given by $(x - 1)(y + 1) = 0$

i.e. $xy + x - y - 1 = 0$.

Ex. 79 Obtain the condition that one of the straight lines given by the equation $ax^2 + hxy + by^2 = 0$, may coincide with one of the those given by the equation $a'x^2 + 2h'xy + b'y^2 = 0$.

Sol. Let the equation of the common line be $y = mx$.

Since, this must satisfy both the given equations, therefore we have

$$bm^2 + 2hm + a = 0 \qquad ...(1)$$

and $$b'm^2 + 2h'm + a' = 0 \qquad ...(2)$$

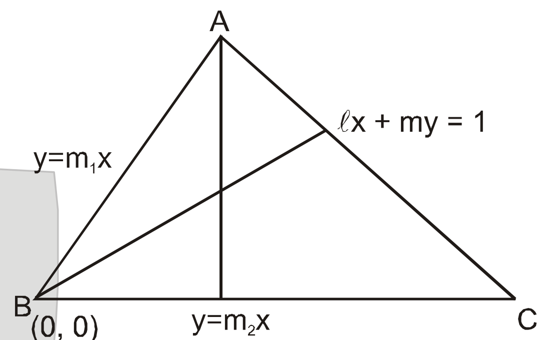
Solving equations (1) and (2), we have

$$\frac{m^2}{2(ha' - h'a)} = \frac{m}{(ab' - a'b)} = \frac{1}{2(bh' - b'h)}$$

Eliminating m, we have $(ab' - a'b)^2 = 4\,(ha' - h'a)\,(bh' - b'h)$.

Ex. 80 Show that the orthocentre of the triangle formed by the straight lines, $a x^2 + 2 h x y + b y^2 = 0$ and $l x + m y = 1$ is a point (x', y') such that,

$$\frac{x'}{l} = \frac{y'}{m} = \frac{a+b}{am^2 - 2hlm + bl^2}.$$



Sol. $a x^2 + 2 h x y + b y^2 \equiv b\,(y - m_1 x)\,(y - m_2 x)$

$\therefore\ m_1 + m_2 = -\frac{2h}{b}$ & $m_1 m_2 = \frac{a}{b}$

The line $l x + m y = 1$ cuts $y - m_1 x = 0$

where $\left.\begin{aligned} x &= \frac{1}{\ell + m m_1} \\ y &= \frac{m_1}{\ell + m m_1} \end{aligned}\right]$ point A.

The equation of the line through this point perpendicular to $y - m_2 x = 0$ is

$$y - \frac{m_1}{\ell + m m_1} = -\frac{1}{m_2}\left(x - \frac{1}{\ell + m m_1}\right)$$

$$(l + m m_1)\,x - 1 + m_2\left\{y\,(\ell + m m_1) - m_1\right\} = 0 \qquad \text{— (1)}$$

The orthocentre lies on the line and also on the line through the origin perpendicular to,

$l x + m y = 1$ i.e. $m x - l y = 0$

or $\frac{x}{\ell} = \frac{y}{m} = \lambda$ say — (2)

put $x = \lambda\, l$ and $y = m\,\lambda$ in (1)

$$\lambda\left\{\ell^2 + \ell m\,(m_1 + m_2) + m^2 . m_1 m_2\right\} = 1 + m_1 m_2.$$

Putting $m_1 + m_2$ & $m_1 m_2$ we get,

$$\lambda = \frac{a+b}{b\ell^2 - 2h\ell m + am^2} \qquad \therefore\ \frac{x'}{\ell} = \frac{y'}{m} = \frac{a+b}{b\ell^2 - 2h\ell m + am^2}$$

Ex. 81 Prove that two straight lines represented by the equation, $a y^4 + b xy^3 + c x^2y^2 + d x^3y + a x^4 = 0$ will bisect the angle between the other two if $c + 6a = 0$ and $b + d = 0$.

Sol. Let $y = mx$. Hence,

$$am^4 + bm^3 + cm^2 + dm + a = 0$$

$$\Sigma m_1 = -\frac{b}{a}\ ;\ \Sigma m_1 m_2 = \frac{c}{a} \qquad \Sigma m_1 m_2 m_3 = -\frac{d}{a}.$$

Note that $m_1 = \tan\theta$; $m_2 = -\cot\theta$; $m_3 = \tan\left(\frac{\pi}{4}+\theta\right)$; $m_4 = \tan\left(\frac{3\pi}{4}+\theta\right)$

$\Rightarrow m_1 m_2 = -1$ and $m_3 m_4 = -1$

Consider $\Sigma m_1 + \Sigma m_1 m_2 m_3 = 0$ $\qquad \Rightarrow -\frac{b}{a} - \quad = 0 \Rightarrow b + d = 0$

again $\Sigma m_1 m_2 = m_1 m_2 + m_1 m_3 + m_1 m_4 + m_2 m_3 + m_2 m_4 + m_3 m_4$

$= -2 + m_1 (m_3 + m_4) + m_2 (m_3 + m_4)$

$= (m_1 + m_2)(m_3 + m_4) - 2$

Substituting the values of m_1, m_2, m_3 and m_4 it simplifies to -6

$$\Rightarrow \frac{c}{a} = -6 \Rightarrow c + 6a = 0$$

Ex. 82 Prove that the general equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, represents two parallel straight lines if $h^2 = ab$ and $bg^2 = af^2$. Prove also that the distance between them is $2\sqrt{\frac{g^2 - ac}{a(a+b)}}$

Sol. The given equation is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad ...(1)$$

Let the two equations represented by (1) be the respectively.

$$(\sqrt{a}.x + \sqrt{b}.y + l) = 0 \quad ...(2)$$

and $$(\sqrt{a}.x + \sqrt{b}.y + m) = 0. \quad ...(3)$$

Then the combined equation will be

$$(\sqrt{a}.x + \sqrt{b}.y + l)(\sqrt{a}.x + \sqrt{b}.y + m) = 0$$

which is identical to (1) and as the coefficients of x^2 and y^2 are equal. Hence equating different coefficients

$$2h = 2\sqrt{(ab)} \quad ...(4)$$

$$2g = \sqrt{a}\,(l + m) \quad ...(5)$$

$$2f = \sqrt{b}\; l + m) \quad ...(6)$$

and $$c = l\,m \quad ...(7)$$

By (4), we get $\quad h^2 = ab$

Dividing (5) by (6), we get $\frac{g}{f} = \frac{\sqrt{a}}{\sqrt{b}}$ or $\frac{g^2}{f^2} = \frac{a}{b}$.

Hence $\quad bg^2 = af^2$. $\qquad$ Proved

Again, if p and p′ be the lengths of perpendiculars from origin on (2) and (3), then the distance between them is $p - p'$.

So $p - p' = \frac{\ell}{\sqrt{(a+b)}} - \frac{m}{\sqrt{(a+b)}} = \frac{(\ell - m)}{\sqrt{(a+b)}} = \sqrt{\left\{\frac{(\ell+m) - 4\ell m}{a+b}\right\}}$

Putting the values of $(l + m)$ and $l\,m$ from (5) and (7), we get

$$p - p' = \sqrt{\left[\frac{(2g/\sqrt{a})^2 - 4c}{a+b}\right]} = 2\sqrt{\left[\frac{g^2 - ac}{a(a+b)}\right]}. \quad \text{Proved}$$

P. HOMOGENISATION

The joint equation of a pair of straight lines joining origin to the points of intersection of the line given by $lx + my + n = 0$................ **(i)** and

the 2nd degree curve : $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$....... **(ii)**

is $ax^2 + 2hxy + by^2 + 2gx\left(\frac{lx+my}{-n}\right) + 2fy\left(\frac{lx+my}{-n}\right) + c\left(\frac{lx+my}{-n}\right)^2 = 0$...... **(iii)**

(iii) is obtained by homogenising (ii) with the help of **(i)**, by writing **(i)** in the form :

$$\left(\frac{lx+my}{-n}\right) = 1.$$

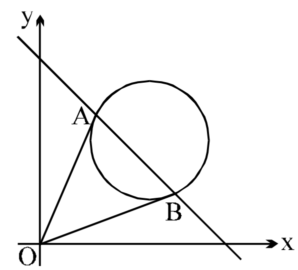
Note :

(i) Equation of any curve passing through the points of intersection of two curves $C_1 = 0$ and $C_2 = 0$ is given by $\lambda C_1 + \mu C_2 = 0$ where λ & μ are parameters.

(ii) Any second degree curve through the four point of intersection of $f(x\,y) = 0$ & $xy = 0$ is given by $f(x\,y) + \lambda xy = 0$ where $f(xy) = 0$ is also a second degree curve.

Ex. 83 Find the equation to the straight lines joining the origin to the points of intersection of the straight line $\frac{x}{a} + \frac{y}{b} = 1$ and the circle $5(x^2 + y^2 + bx + ay) = 9ab$. Also find the linear relation between *a* and *b* so that these straight lines may be at right angle.

Sol. Homogenising, $5(x^2 + y^2) + 5(bx + ay)\left(\frac{x}{a} + \frac{y}{b}\right) - 9ab\left(\frac{x}{a} + \frac{y}{b}\right)^2 = 0$



since lines are perpendicular
hence coefficient of $(x^2 + y^2) = 0$

$$10 + 5\left(\frac{b}{a} + \frac{a}{b}\right) - 9\left(\frac{b}{a} + \frac{a}{b}\right) = 0 \quad \Rightarrow \quad 10 = 4\left(\frac{a^2 + b^2}{ab}\right)$$

$\Rightarrow \quad 4a^2 + 4b^2 = 10ab \quad \Rightarrow \quad 2(a^2 + b^2) = 5ab \quad \Rightarrow \quad 2a^2 - 5ab + 2b^2 = 0$

$\Rightarrow \quad 2a^2 - 4ab - ab + 2b^2 = 0 \quad \Rightarrow \quad 2a(a - 2b) - b(a - 2b) = 0$

$\Rightarrow \quad a = 2b$ or $2a = b$

Ex. 84 Prove that the straight line joining the origin to the points of intersection of the straight line $kx + hy = 2hk$ with the curve
$(x - h)^2 + (y - k)^2 = c^2$ are at right angles if, $h^2 + k^2 + c^2$.

Sol. The given line is $kx + hy = 2hk$ or $\frac{x}{2h} + \frac{y}{2k} = 1$. ...(1)

The given curve is $(x - h)^2 - (y - k)^2 = c^2$

or $x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - c^2 = 0$. ...(2)

Making (2) homogeneous with the help of (1), we get

$$x^2 + y^2 - 2hx\left(\frac{x}{2h} + \frac{y}{2k}\right) - 2ky\left(\frac{x}{2h} + \frac{y}{2k}\right) + (h^2 + k^2 - c^2)\left(\frac{x}{2h} + \frac{y}{2k}\right)^2 = 0 \quad ...(3)$$

Coefficient of x^2 in (3) is $= \frac{h^2 + k^2 - c^2}{4h^2}$ Coefficient of y^2 in (3) is $= \frac{h^2 + k^2 - c^2}{4k^2}$

If the lines represented by (3) are perpendicular to each other, then

$$\frac{h^2 + k^2 - c^2}{4h^2} + \frac{h^2 + k^2 - c^2}{4k^2} = 0 \quad \text{or} \quad (h^2 + k^2 - c^2)\left(\frac{1}{4h^2} + \frac{1}{4k^2}\right) = 0.$$

As $\left(\frac{1}{4h^2} + \frac{1}{4k^2}\right)$ cannot be zero, being sum of two squares, hence

$h^2 + k^2 - c^2 = 0$ or $h^2 + k^2 = c^2$. Proved.

Ex. 85 Prove that the angle between the lines joining the origin to the points of intersection of the straight line $y = 3x + 2$ with the curve $x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0$ is $\tan^{-1}\frac{2\sqrt{2}}{3}$.

Sol. Equation of the given curve is $x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0$

and equation of the given straight line is $y - 3x = 2$; $\therefore$ $\frac{y-3x}{2} = 1$

Making equation (1) homogeneous equation of the second degree in x and y with the help of (1), we have

$$x^2 + 2xy + 3y^2 + 4x\left(\frac{y-3x}{2}\right) + 8y\left(\frac{y-3x}{2}\right) - 11\left(\frac{y-3x}{2}\right)^2 = 0$$

or $\quad x^2 + 2xy + 3y^2 + \frac{1}{2}(4xy + 8y^2 - 12x^2 - 24xy) - \frac{11}{4}(y^2 - 6xy + 9x^2) = 0$

or $\quad 4x^2 + 8xy + 12y^2 + 2(8y^2 - 12x^2 - 20xy) - 11(y^2 - 6xy + 9x^2) = 0$

or $\quad -119x^2 + 34xy + 17y^2 = 0$ or $119x^2 - 34xy - 17y^2 = 0$

or $\quad 7x^2 - 2xy - y^2 = 0$

This is the equation of the liens joining the origin to the points of intersection of (1) and (2).
comparing equation (3) with the equation $ax^2 + 2hxy + by^2 = 0$
we have $a = 7$, $b = -1$ and $2h = -2$ i.r. $h = -1$
If θ be the acute angle between pair of lines (3), then

$$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right| = \left|\frac{2\sqrt{1+7}}{7-1}\right| = \frac{2\sqrt{8}}{6} = \frac{2\sqrt{2}}{3}$$

$\therefore \quad \theta = \tan^{-1}\frac{2\sqrt{2}}{3}$ proved

Ex. 86 If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, represent a pair of straight lines, prove that the quation to the third pair of straight lines passing through the points where these meet the axes is

$$ax^2 - 2hxy + by^2 + 2gx + 2fy + c + \frac{4fg}{c}xy = 0$$

Sol. We are given the equation as

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad ...(1)$$

The equation of y–axis is x = 0 and x–axis is y = 0. Hence the combined equation will be $xy = 0$. ...(2)

Equation of the curve passing through the point of intersection of (1) and (2) will be

$ax^2 + 2hxy + by^2 + 2gx + 2fy + 2\lambda xy = 0$

or $\quad ax^2 + 2(h + \lambda)xy + by^2 + 2gx + 2fy + c = 0.$ ...(3)

If (3) represent two straight lines, then its discriminant must be zero.

So $\quad abc + 2.f.g.(h + \lambda) - af^2 - bg^2 - c(h + \lambda)^2 = 0$

or $\quad abc + 2fgh - af^2 - bg^2 - ch^2 + 2\lambda(fg - ch) - c\lambda^2 = 0$ ...(4)

As (1) represents two straight line

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$

Putting in (4), we get $2\lambda(fg - ch) - c\lambda = 0$ or $\lambda = \frac{2(fg-ch)}{c}$.

Putting in (3), we get $ax^2 + 2\left[h + \frac{2(fg-ch)}{c}\right]xy + by^2 + 2gx + 2fy + c = 0$

or $\qquad ax^2 + \dfrac{4\,fg - 2\,ch}{c}\,xy + by^2 + 2gx + 2dy + c = 0$

or $\qquad ax^2 - 2hxy + by^2 + 2gx + 2fy + c + \dfrac{4\,fg}{c}\,xy = 0.$ $\qquad$ Proved

Q. TRANSFORMATION OF AXES

Ex. 87 What does the equation $3x^2 + 2xy + 3y^2 - 18x - 22y + 50 = 0$ become when referred to rectangular axes through the point (2, 3), the new axis of x making an angle of 45° with the old ?

Sol. First change the origin, by putting $x' + 2$, $y' + 3$ for x, y respectively.

The new equation will be

$3(x' + 2)^2 + 2\,(x' + 2)\,(y' + 3) + 3(y' + 3)^2 - 18\,(x' + 2) - 22\,(y' + 3) + 50 = 0;$

which reduces to $\qquad 3x'^2 + 2x'y' + 3y'^2 - 1 = 0,$

or, suppressing the accents, to

$$3x^2 + 2xy + 3y^2 = 1 \quad \text{.................................(i)}$$

To turn the axes through an angle of 45° we must write $x'\dfrac{1}{\sqrt{2}} - y'\dfrac{1}{\sqrt{2}}$

for x, and $x'\dfrac{1}{\sqrt{2}} + y'\dfrac{1}{\sqrt{2}}$ for y. Equation (i) will then be

$$3\left(\frac{x' - y'}{\sqrt{2}}\right)^2 + \frac{x' - y'}{\sqrt{2}}\,.\,\frac{x' - y'}{\sqrt{2}} + 3\left(\frac{x' + y'}{\sqrt{2}}\right)^2 = 1,$$

which reduces to $4x'^2 + 2y'^2 = 1$.

Thus the required equation is $4x^2 + 2y^2 = 1$.

Ex. 88 Find the new abscissa and ordinate if the straight lines $2x - 3y - 12 = 0$, $3x + 2y - 4 = 0$ are the new axes of x and y respectively.

Sol. The new abscissa = the perpendicular from (x, y) upon the new

axis of y, $(3x + 2y - 4 = 0)$ $\qquad = \dfrac{3x + 2y - 4}{\sqrt{13}}$.

The new ordinate = the perpendicular from (x, y) upon the new

axis of x, $(2x - 3y - 12 = 0)$ $\qquad = \dfrac{2x - 3y - 12}{\sqrt{13}}$

Ex. 89 Transform the equation of the curve $\dfrac{(3x + 4y)^2}{25} - \dfrac{(4x - 3y)^2}{50} = 1$,

if we make $\quad 3x + 4y = 0$ the new axis of y, and $4x - 3y = 0$ the new axis of x.

Sol. The new abscissa = $\dfrac{3x + 4y}{5}$,

and $\qquad$ the new ordinate = $\dfrac{4x - 3y}{5}$

$\therefore$ $\quad$ the transformed equation is $\dfrac{(5x)^2}{25} - \dfrac{(5y)^2}{50} = 1$, $\quad$ or $\quad x^2 - \dfrac{y^2}{2} = 1.$

Ex. 90 Transform to parallel axes through the point (1, –2) the equations (1) $y^2 - 4x + 4y + 8 = 0$. and (2) $2x^2 + y^2 - 4x + 4y = 0$.

Sol.(i) The equation is $y^2 - 4x + 4y + 8 = 0$. The origin is transfered to (1, –2). So the new equation will be

$$(y' - 2)^2 - 4(x' + 1) + 4(y' - 2) + 8 = 0$$

or $$y'^2 - 4y' + 4 - 4x' - 4 + 4y' - 8 + 8 = 0$$

or $$y'^2 = 4x'.$$

(ii) The equation is $2x^2 + y^2 - 4x + 4y = 0$.

Transferring the origin to (1, –2), we get

$$2(x' + 1)^2 + (y' - 2)^2 - 4(x' + 1) + 4(y' - 2) = 0$$

or $$2x'^2 + 4x' + 2 + y'^2 - 4y' + 4 - 4x' - 4 + 4y' - 8 = 0$$

or $$2x'^2 + y'^2 = 6.$$

Ex. 91 By transforming to parallel axes through a properly chosen point (h, k) Prove that the equation $12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0$, can be reduced to one containing only terms of the second degree.

Sol. The given equation is $12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0$.

Let the origin be transferred to (h, k) axes being parallel to the previous axes; then the equation becomes

$$12(x' + h)^2 - 10(x' + h)(y' + k) + 2(y' + k)^2 + 11(x' + h) - 5(y' + k) + 2 = 0$$

or $$12x'^2 + 12h^2 - 24x'h - 10x'y' - 10x'k - 10y'h - 10hk + 2y'^2$$
$$+ 2k^2 + 4y'k + 11x' + 11h - 5y' - 5k + 2 = 0 \quad ....(1)$$

or $$12x'^2 + 2y'^2 - 10x'y' + x'(24h - 10k + 11) + y'(-10 + 4k - 5)$$
$$+ 12h^2 - 10hk + 2k^2 + 11h - 5k + 2 = 0.$$

If this equation contains the terms of x^2 and y^2 and constant terms only, then the coefficients of x' and y' must be zero.

So $$24h - 10k + 11 = 0 \quad ....(2)$$

and $$(-10h + 5k - 5) = 0.$$

Solving (2) and (3), we get $h = -\dfrac{3}{2}$ and $k = -\dfrac{5}{2}$

Hnece the required point is $\left(-\dfrac{3}{2}, -\dfrac{5}{2}\right)$.

If we subsitute these values in (1), the equation reduces to

$$12x'^2 - 10x'y' - 2y'^2 = 0.$$