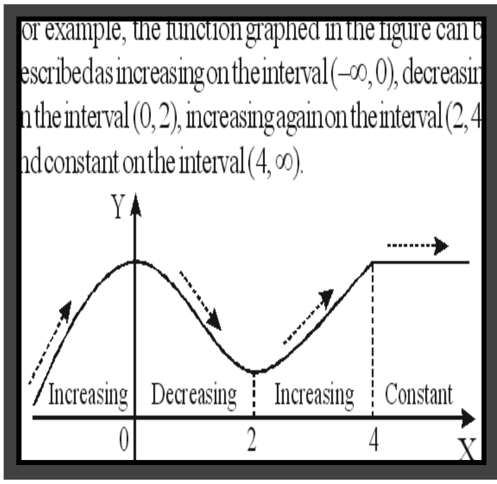


CHAPTER 6

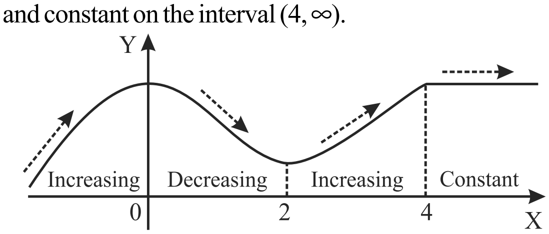
MONOTONICITY

6.1 DEFINITIONS

Introduction

Monotonicity is the study of increasing–decreasing behaviour of a function. The terms increasing, decreasing, and constant are used to describe the behaviour of a function over an interval as we travel left to right along its graph.

For example, the function graphed in the figure can be described as increasing on the interval $(-\infty, 0)$, decreasing on the interval $(0, 2)$, increasing again on the interval $(2, 4)$, and constant on the interval $(4, \infty)$.



Monotonicity about a point

Let a function f be defined on an open interval containing the point $x = a$. We have a set of four standard terms to describe the increasing–decreasing behaviour of the function in a sufficiently small neighbourhood around $x = a$. They are as follows:

(i) Strictly increasing

(ii) Strictly decreasing

(iii) Non-decreasing

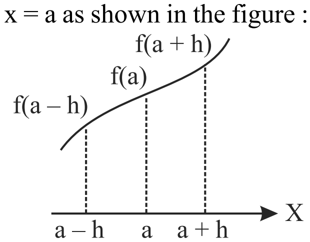
(iv) Non-increasing

If a function follows any of the four conditions, it is said to be monotonic about $x = a$, otherwise, it is said to be non-monotonic.

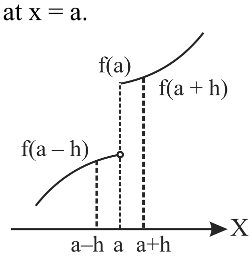
(i) Strictly increasing

A function $f(x)$ is said to be strictly increasing about the point $x = a$ if $f(a - h) < f(a) < f(a + h)$, where h is a small positive arbitrary number.

Consider the graph of a function in the neigh-bourhood of the point $x = a$ as shown in the figure :



We notice that $f(a - h)$ is less than $f(a)$ while $f(a + h)$ is greater than $f(a)$ for any h in a small neighbourhood around $x = a$. Then, we say that the function is strictly increasing about $x = a$. A similar situation is found in the graph of the following figure, which is discontinuous at $x = a$.



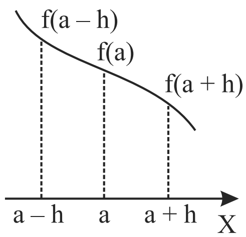
For example, the functions $f(x) = e^x$, $f(x) = 2x + 1$, and

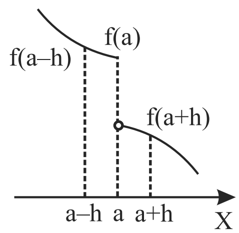
$$f(x)=\begin{cases} 1-x^2, & x<0, \\ 2+x^2, & x \geq 0 \end{cases}$$ are strictly increasing about $x = 0$.

(ii) Strictly decreasing

A function f(x) is said to be strictly decreasing about the point $x = a$ if $f(a - h) > f(a) > f(a + h)$, where h is a small positive arbitrary number.

Consider the graph of a function in the neighbourhood of the point $x = a$ as shown in the figures below :



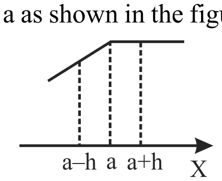


Each of these functions are strictly decreasing about $x = a$.

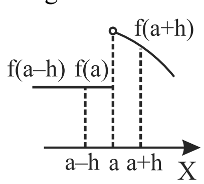
(iii) Non-decreasing

A function f(x) is said to be non-decreasing about the point $x = a$ if $f(a - h) \leq f(a) \leq f(a + h)$, where h is a small positive arbitrary number.

Consider the graph of a function in the neighbourhood of the point $x = a$ as shown in the figure :



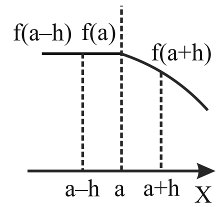
We observe that in the given figure,
$f(a - h) < f(a) = f(a + h)$, hence, we say that the function is non-decreasing at $x = a$. The function shown below is also non-decreasing at $x = a$.

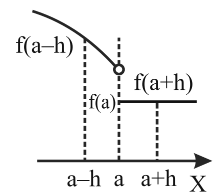


(iv) Non-increasing

A function f(x) is said to be non-increasing about the point $x = a$ if $f(a - h) \geq f(a) \geq f(a + h)$, where h is a small positive arbitrary number.

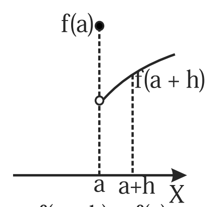
The functions shown below are non-increasing at $x = a$.





Note:

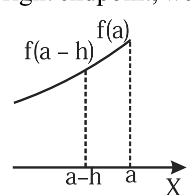
1. If f(x) is constant in the neighbourhood of the point $x = a$ then it said to either non-decreasing or non-increasing.
2. If $x = a$ is an endpoint then we use the appropriate one-sided inequality to test monotonicity of f(x) at $x = a$. For example, if $x = a$ is the left endpoint, we check as shown below :



$f(a + h) < f(a)$

strictly decreasing at x = a

If $x = a$ is the right endpoint, we check as follows:



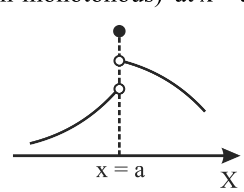
$f(a) > f(a - h)$

strictly increasing at x = a

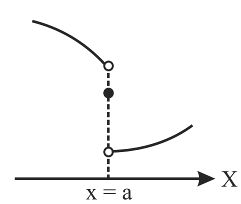
3. It should be noted that we can talk of monotonicity of f at $x = a$ only if $x = a$ lies in the domain of f, without any consideration of continuity or differentiability of f at $x = a$.

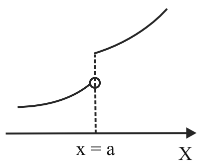
Example 1. Which of the following function is strictly increasing, strictly decreasing, non-increasing, non-decreasing or neither increasing nor decreasing (non-monotonous) at $x = a$.

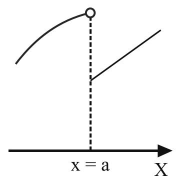
(i)

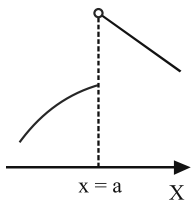


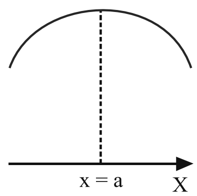
(ii)

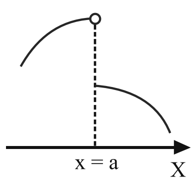


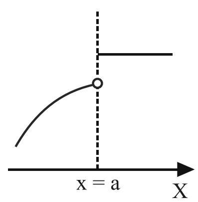
(iii)


(iv)


(v)


(vi)


(vii)


(viii)


Solution

(i) Neither increasing nor decreasing as $f(a-h) < f(a)$ and $f(a) > f(a+h)$

(ii) Strictly decreasing as $f(a-h) > f(a) > f(a+h)$

(iii) Strictly increasing as $f(a-h) < f(a) < f(a+h)$

(iv) Neither increasing nor decreasing as $f(a-h) > f(a)$ and $f(a) < f(a+h)$

(v) Strictly increasing

(vi) Neither increasing nor decreasing as $f(a-h) < f(a)$ and $f(a) > f(a+h)$

(vii) Strictly decreasing

(viii) Non-decreasing as $f(a-h)<f(a)$ and $f(a)=f(a+h)$.

Example 2. Examine the behaviour of the function $f(x)=\dfrac{1}{1+x}$ at the point $x=0$.

Solution We have $f(x)=\dfrac{1}{1+x}$.

$f(0)=1$

$$f(0-h)=\frac{1}{1-h}>1$$

$$f(0+h)=\frac{1}{1+h}<1$$

Since, $f(0-h) > f(0) > f(0+h)$, where h is a small positive arbitrary number, f is strictly decreasing at $x=0$.

Test for finding Monotonicity at a point

Sufficient Conditions for Monotonicity at a point

Let a function f be differentiable at $x = a$.

(i) If $f'(a) > 0$ then $f(x)$ is strictly increasing at $x = a$.

(ii) If $f'(a) < 0$ then $f(x)$ is strictly decreasing at $x = a$.

(iii) If $f'(a) = 0$ then we need to examine the signs of $f'(a-h)$ and $f'(a+h)$.

 (a) If $f'(a-h) > 0$ and $f'(a+h) > 0$ then $f(x)$ is strictly increasing at $x = a$.

 (b) If $f'(a-h) < 0$ and $f'(a+h) < 0$ then $f(x)$ is strictly decreasing at $x = a$.

 (c) If $f'(a-h)$ and $f'(a+h)$ have opposite signs then $f(x)$ is neither increasing nor decreasing (non-monotonous) at $x = a$.

(iv) If none of the above conditions are followed, then the function needs more investigation, which shall be discussed later.

Example 3. Examine the behaviour of the function $f(x) = x^3 - 3x + 2$ at the points $x = 0, 1, 2$.

Solution $f(x)=x^3-3x+2$

$f'(x)=3(x^2-1)$

At the point $x = 0$, $f'(0) = -3 < 0$

$\Rightarrow$ $f(x)$ is decreasing at $x = 0$.

At the point $x = 1$, $f'(1) = 0$

But, $f'(1-h)$ = negative and $f'(1+h)$ = positive

$\Rightarrow$ $f(x)$ is neither increasing nor decreasing at $x = 1$.

At the point $x = 2$, $f'(2) = 9 > 0$

$\Rightarrow$ $f(x)$ is increasing at $x = 2$.

Example 4. Let $f(x)=\begin{cases}(x-1)e^x+1, & x<0\\ -(1+x^{4/3}), & x\ge 0\end{cases}$.

Investigate the behaviour of the function at $x = 0$.

Solution We have

$$f'(x)=\begin{cases}xe^x, & x<0\\ -\dfrac{4}{3}x^{1/3}, & x>0\end{cases}$$

At $x=0$, $f'(0^-)=0$ and $f'(0^+)=0$.

But $f'(0-h)<0$ and $f'(0+h)<0$

Hence $f(x)$ is strictly decreasing at $x=0$.

Example 5. Test the function $f(x) = 1 - (x-2)^{3/5}$ for monotonicity at $x = 2$.

Solution We have $f'(x) = -\frac{2}{5}(x-2)^{-2/5}$.

$f'(2^-) = -\infty$ and $f'(2^+) = -\infty$

$f(x)$ is continuous at $x = 2$.

The function has an infinite(negative) derivative at $x = 2$. This implies that $f(x)$ is strictly decreasing at $x = 2$.

Test for finding monotonicity at an endpoint

If $x = a$ is an endpoint then we use the sign of the appropriate one-sided derivative to test the monotonocity of $f(x)$ at $x = a$. Assume that the function f is differentiable at $x = a$.

If $x = a$ is the left endpoint, we check as follows :

(i) If $f'(a^+) > 0$, then $f(x)$ is strictly increasing at $x = a$.

(ii) If $f'(a^+) < 0$, then $f(x)$ is strictly decreasing at $x = a$.

(iii) If $f'(a^+) = 0$, but $f'(a + h) > 0$, then $f(x)$ is strictly increasing at $x = a$.

(iv) If $f'(a^+) = 0$, but $f'(a + h) < 0$, then $f(x)$ is strictly decreasing at $x = a$.

If $x = a$ is the right endpoint, we check as follows:

(i) If $f'(a^-) > 0$, then $f(x)$ is strictly increasing at $x = a$.

(ii) If $f'(a^-) < 0$, then $f(x)$ is strictly decreasing at $x = a$.

(iii) If $f'(a^-) = 0$, but $f'(a - h) > 0$, then $f(x)$ is strictly increasing at $x = a$.

(iv) If $f'(a^-) = 0$, but $f'(a - h) < 0$, then $f(x)$ is strictly decreasing at $x = a$.

For example, consider the function $f(x) = (x-1)^{3/2}$.

$x = 1$ is the left endpoint of the domain $[1, \infty)$.

$$f'(x) = \frac{3}{2}(x-1)^{1/2}$$

$$f'(1^+) = 0, \text{ but } f'(1+h) = \frac{3}{2}h^{1/2} > 0.$$

Hence, $f(x)$ is strictly increasing at $x = 1$.

Example 6. Let $f(x) = \begin{cases} x^2 + 2x, & -2 \le x < 0 \\ \sin^{-1} x, & 0 \le x \le 1 \end{cases}$.

Investigate the behaviour of the function at $x = -2$, 0 and 1.

Solution We have

$$f'(x) = \begin{cases} 2x + 2, & -2 \le x < 0 \\ \dfrac{1}{\sqrt{1-x^2}}, & 0 < x < 1 \end{cases}$$

At $x = -2$, $f'(-2^+) < 0$. Hence $f(x)$ is strictly decreasing at $x = -2$.

At $x = 0$, $f'(0^-) = 2 > 0$, $f'(0^+) = 1 > 0$.

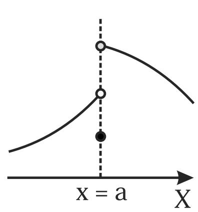
Hence $f(x)$ is strictly increasing at $x = 0$.

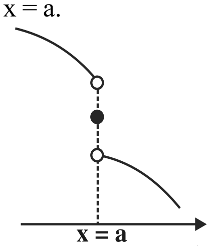
At $x = 1$, $f'(1^-) = \infty$. f is continuous at $x = 1$.

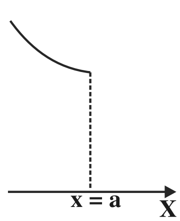
The infinite(positive) derivative implies that, $f(x)$ is strictly increasing at $x = 1$.

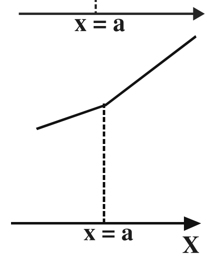
Concept Problems A

1. For each of the following graph comment whether f(x) is increasing or decreasing or neither increasing nor decreasing at x = a.

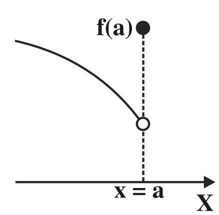
(i) 

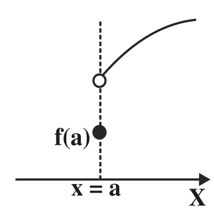
(ii) 

(iii) 

(iv) 

2. Consider the following graphs of functions which have x = a as an endpoint. Find the monotonicity at x = a.

(i) 

(ii) 

3. Let $f(x) = x^3 - 3x^2 + 3x + 4$. Comment on the monotonic behaviour of f(x) at (i) $x = 0$ (ii) $x = 1$.

4. Find out the behaviour of the function $y = x - \ln x$ at the points $x_1 = 1/2$, $x_2 = 2$, $x_3 = e$ and $x_4 = 1$, and show that if the given function increases at the point $x = a > 0$, then it decreases at the point $x = \frac{1}{a}$.

5. Find the behaviour of the functions at $x = 0$:
   (i) $y = x^5 - x^3$ (ii) $y = |ln(x+1)|$
   (iii) $y = 1 - x^{4/5}$
6. Show that the function $y = ln\ (x^2 + 2x - 3)$ increases at the point $x_1 = 2$ and decreases at the point $x_2 = -4$.
7. Draw the graph of function $f(x) = \begin{cases} x & 0 \le x \le 1 \\ [x] & 1 \le x \le 2 \end{cases}$.
   Comment on the monotonic behaviour of $f(x)$ at $x = 0, 1$ and 2.
8. Test the behaviour of $f(x) = x\{x\}$ at $x = 0$, where $\{.\}$ represents fractional part function.
9. Comment on the monotonic behaviour of the following functions at $x = 1$.
   (i) $f(x) = x^2(x-2)^2$ (ii) $f(x) = x\,ln\,x$
   (iii) $f(x) = \sin x + \cos x$.

6.2 MONOTONICITY OVER AN INTERVAL

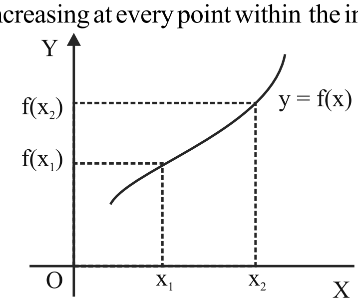
Let a function f be defined on a domain D. If a function follows any of the four conditions given below, it is said to be monotonic in D, otherwise, it is said to be non-monotonic in D.

(i) Strictly increasing function

The function f(x) is said to be strictly increasing on D if for every two points x_1 and x_2 belonging to D and satisfying the inequality $x_1 < x_2$ the inequality

$f(x_1) < f(x_2)$ holds true.

It means that there is a definite increase in the value of f(x) with an increase in the value of x (See the figure given below). In other words, if the graph of a function is rising on an interval (and never flattens out on that interval), we say that f is strictly increasing on that interval. Note that a function is strictly increasing in an interval (a, b) if it is strictly increasing at every point within the interval.



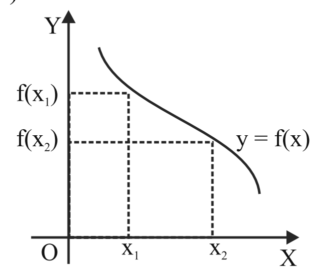
Note: In different texts, strictly increasing function is named differently. For instance, it is called as monotonically increasing, strictly monotonically increasing, steadily increasing, increasing, etc.

(ii) Strictly decreasing function

The function f(x) is said to be strictly decreasing on D if for every two points x_1 and x_2 belonging to D and satisfying the inequality $x_1 < x_2$ the inequality

$f(x_1) > f(x_2)$ holds true.

It means that there is a definite decrease in the value of f(x) with an increase in the value of x (See the figure given below).

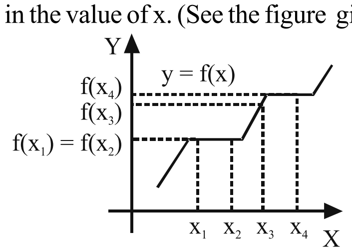


(iii) Non-decreasing function

The function f(x) is said to be non-decreasing on D if for every two points x_1 and x_2 belonging to D and satisfying the inequality $x_1 < x_2$ the inequality

$f(x_1) \le f(x_2)$ holds true.

It means that the value of f(x) never decreases with an increase in the value of x. (See the figure given below).



In the above figure we find that for $x_1 < x_2 < x_3 < x_4$ we have $f(x_1) = f(x_2) < f(x_3) < f(x_4)$. Such a function is called non-decreasing.

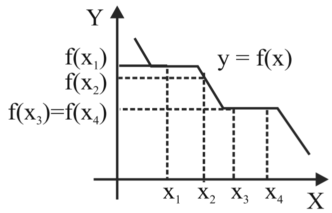
Note: In some texts, non-decreasing function is named as increasing.

(iii) Non-increasing function

The function f(x) is said to be non-increasing on D if for every two points x_1 and x_2 belonging to D and satisfying the inequality $x_1 < x_2$ the inequality

$f(x_1) \ge f(x_2)$ holds true.

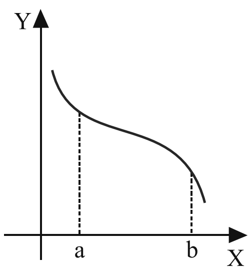
It means that the value of f(x) never increases with an increase in the value of x. (See the figure given below).



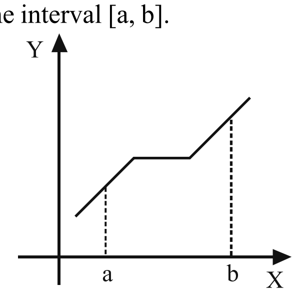
In the above figure we find that for $x_1 < x_2 < x_3 < x_4$ we have $f(x_1) > f(x_2) > f(x_3) = f(x_4)$. Such a function is called non-increasing.

Note: A function which is constant over an interval, is said to be either non-decreasing or non-increasing.

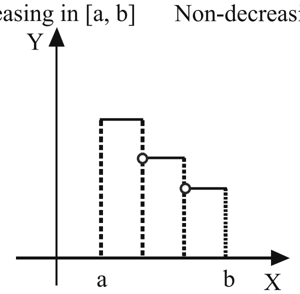
The following figures show the behaviour of some functions monotonic in the interval [a, b].



Strictly decreasing in [a, b]

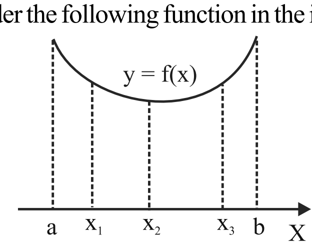


Non-decreasing in [a, b]



Non-increasing in [a, b]

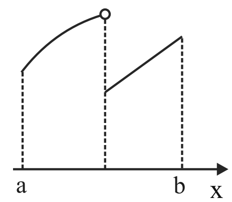
Now consider the following function in the interval [a, b].



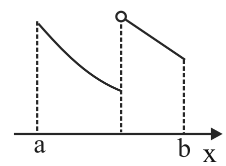
Here, we find that for $x_1 < x_2 < x_3$, $f(x_1) > f(x_2) < f(x_3)$. So, the order between function's value at two different points is not maintained throughout the interval. Hence, the function is not monotonic in the interval [a, b].

Note that the following functions are not monotonic in the interval [a, b].

(i)

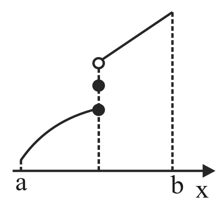


(ii)

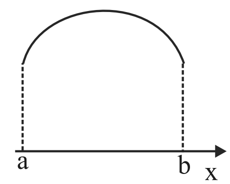


Example 1. Which of the following function is strictly increasing, strictly decreasing, non-increasing, non-decreasing or neither increasing nor decreasing (non-monotonous) in the interval [a, b] ?

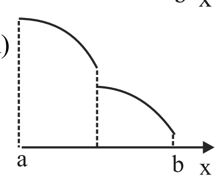
(i)



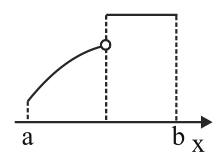
(ii)



(vii)



(viii)



Solution

(i) Strictly increasing as $f(x_1) < f(x_2)$ for every two points x_1 and x_2 belonging to the interval [a, b] and satisfying $x_1 < x_2$.

(ii) Neither increasing nor decreasing

(iii) Strictly decreasing as $f(x_1) > f(x_2)$ for every two points x_1 and x_2 belonging to the interval [a, b] and satisfying $x_1 < x_2$.

(iv) Non-decreasing as $f(x_1) \le f(x_2)$ for every two points x_1 and x_2 belonging to the interval [a, b] and satisfying $x_1 < x_2$.

Test for finding monotonicity over an interval

Let a function f be defined and continuous on a certain interval (a, b) and have a derivative everywhere in the interval (a, b) except possibly at a finite number of points. We can predict the behaviour of f in the interval by studying the sign of its derivative $f'(x)$ over the interval.

Necessary Conditions for Monotonicity

(i) If a differentiable function f(x) increases in an interval its derivative $f'(x)$ is nonnegative: $f'(x) \geq 0$.

(ii) If a differentiable function f(x) decreases in an interval its derivative $f'(x)$ is non-positive: $f'(x) \leq 0$.

(iii) If a differentiable function f(x) does not vary in an interval (i.e. is equal to a constant) its derivative is identically equal to zero : $f'(x) = 0$.

Theorem 1 If a differentiable function f(x) is strictly increasing on the interval (a, b), then $f'(x) \geq 0$ for any x in the interval (a, b).

Proof According to the definition of a function strictly increasing on (a, b), if $x > x_0$, then $f(x) > f(x_0)$. and if $x < x_0$, then $f(x) < f(x_0)$.

Consequently, for any x_0 and x in (a, b), $x \neq x_0$, the inequality $\frac{f(x) - f(x_0)}{x - x_0} > 0$ holds true.

Since f(x) is differentiable on (a, b), proceeding to the limit in the last inequality as $x \to x_0$, we get

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \geq 0$$

The theorem is proved.

Theorem 2 If a differentiable function f(x) is strictly decreasing of the interval (a, b), then $f'(x) \leq 0$ for any x in the interval (a, b).

Proof Since f(x) is a strictly decreasing function, the function $F(x) = -f(x)$ is a strictly increasing one, and therefore, by Theorem 1, $F'(x_0) = -f'(x_0) \geq 0$ for any $x_0 \in$ (a, b). Hence, it follows that $f'(x_0) \leq 0$ for any $x_0 \in$ (a, b). The theorem is proved.

Theorem 3 If a function f(x) is constant in the interval (a, b), then $f'(x) = 0$ for any x in the interval (a, b).

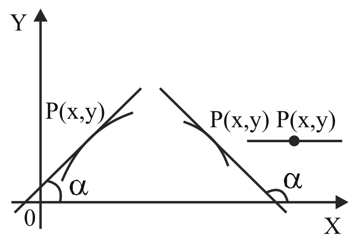
Proof If f(x) is constant its derivative is known to be equal to zero : $f'(x) = 0$.

Note: The foregoing theorems express the following geometric fact. If on an interval (a, b) a function f(x) is strictly increasing, then as the variable point P(x,y) traces the graph of the function from left to right, i.e. as the abscissa increases, the value of the function moves upward along y-axis.

Then, the tangent to the curve y = f(x) at each point on this interval forms an acute angle α with the x-axis (or, at some points, the tangent line is horizontal); the tangent of this angle is nonnegative:

$f'(x) = \tan\alpha \geq 0$.

If the function f(x) is strictly decreasing on the interval (a, b), then the angle of inclination of the tangent line forms on obtuse angle (or, at some points, the tangent line is horizontal) the tangent of this angle is nonpositive.



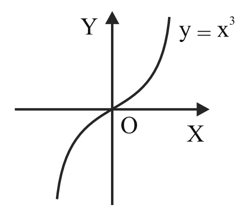
If the variable point P(x,y) moves on a horizontal line then, the tangent is also horizontal and hence,

$f'(x) = \tan 0 = 0$.

It should be stressed that the derivative of a strictly increasing or strictly decreasing function may vanish at some separate points.

For instance, the cubic function $f(x) = x^3$ increases strictly throughout the x-axis, but its derivative $f'(x) = 3x^2$ turns into zero at the point x = 0, although at all the other points it is positive. The function is strictly increasing at x = 0 since larger values of x have correspondingly larger values of f. We can see from the figure that

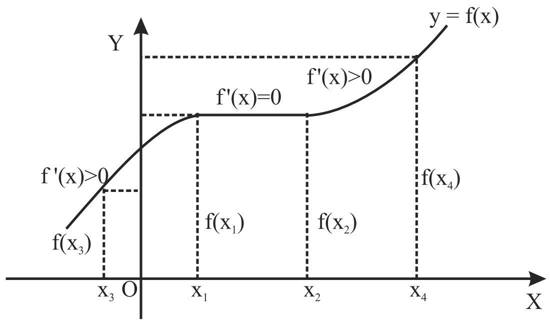
$$f(0 - h) < f(0) \text{ and } f(0 + h) > f(0).$$



Geometrically, this means that the tangent to the graph of a strictly increasing or strictly decreasing function may be parallel to x-axis at some points.

Theorem 4 For the function f(x) differentiable on an interval I, not to decrease (not to increase) on that interval, it is necessary and sufficient that $\forall\, x \in I$ the inequality $f'(x) \geq 0$ $(f'(x) \leq 0)$ be satisfied.

Consider the graph of a non-decreasing function.



It is clear from the figure that

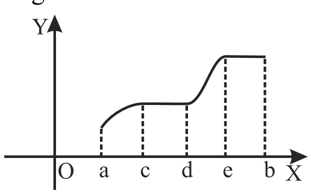
for $x_3 < x < x_1$ we have $f(x) < f(x_1)$ and $f'(x) > 0$

for $x_1 < x < x_2$ we have $f(x_1) = f(x_2)$ and $f'(x) = 0$

for $x_2 < x < x_4$ we have $f(x) > f(x_2)$ and $f'(x) > 0$

Combining all cases, we say that for any $x \in (x_3, x_4)$ we have $f(x_1) \leq f(x_2)$ and $f'(x) \geq 0$.

Now, see the figure shown below :



Here also $f'(x) \geq 0$ for all $x \in (a, b)$, but note that in this case, equality of $f'(x) = 0$ holds for all $x \in (c, d)$ and (e, b). Here $f'(x)$ becomes identically zero on two subintervals and hence the given function cannot be assumed to be strictly increasing in (a, b). We say that f(x) is non-decreasing in (a, b).

For example, $f(x) = x^3 + |x|^3$, $x \in R$ is not a strictly increasing function, rather, it is a non-decreasing function.

Thus, if $f'(x) \geq 0$ in an interval I, with $f'(x) = 0$ on one or more subintervals of I, then f(x) is said to be a non-decreasing function in the interval I.

But, if f′(x) vanishes at a countable number of isolated points, provided it be elsewhere uniformly positive, then f(x) will strictly increase.

Example 2. Prove that $f(x) = x - \sin x$ is a strictly increasing function.

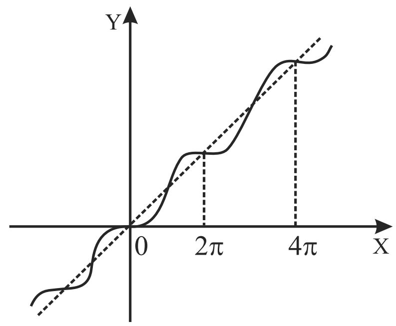
Solution Given $f(x) = x - \sin x$

$\Rightarrow f'(x) = 1 - \cos x$

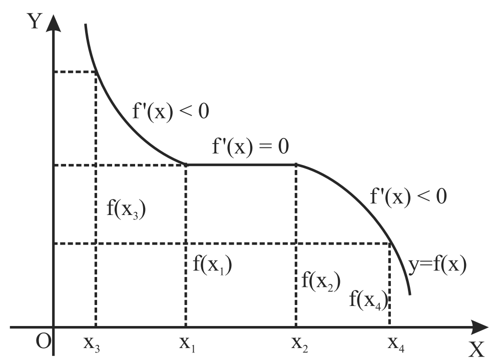
We have $f'(x) \geq 0$.

$f'(x) = 0$ at $x = 0, \pm 2\pi, \pm 4\pi, \ldots\ldots$

Now, $f'(x) > 0$ everywhere except at $x = 0, \pm 2\pi, \pm 4\pi$, ... but all these points are discrete (separated) and do not form an interval. Hence we can conclude that f(x) is strictly increasing for all x. In the figure, we see that the graph passes through the point $(2n\pi, 2n\pi)$, $n \in I$, by increasing continuously, across a horizontal tangent.



Now, consider the graph of a non-increasing function.



It is clear from the figure that

for $x_3 < x < x_1$ we have $f(x) > f(x_1)$ and $f'(x) < 0$

for $x_1 < x < x_2$ we have $f(x_1) = f(x_2)$ and $f'(x) = 0$

for $x_2 < x < x_4$ we have $f(x) < f(x_2)$ and $f'(x) < 0$

Combining all cases, we say that for any $x \in (x_3, x_4)$ we have $f(x_1) \geq f(x_2)$ and $f'(x) \leq 0$.

Thus, if $f'(x) \leq 0$ in an interval I, with $f'(x) = 0$ on one or more subintervals of I, then f(x) is said to be a non-increasing function in the interval I.

But, if $f'(x)$ vanishes at a finite number of isolated points, and is otherwise negative, then f(x) will strictly decrease.

It is now clear that, in an interval of monotonicity of a differentiable function its derivative cannot change sign to the opposite.

The above results allow us to judge upon the sign of the derivative of a monotonous differentiable function in a given interval by its increase or decrease in this interval. But when we begin to investigate a given function, its behaviour is usually not known, and therefore, it is much more important to establish the converse of the above results, which enables us to study the character of the variation of a function in a given interval by reducing the problem to the simpler question of determining the sign of its derivative.

Sufficient Conditions for Monotonicity

Theorem Let f(x) be a differentiable function on the interval (a, b). Then :

(i) If the derivative $f'(x)$ is everywhere positive (i.e. $f'(x) > 0$) in the interval (a, b), then the function f(x) is strictly increasing in the interval (a, b).

(ii) If the derivative $f'(x)$ is everywhere negative (i.e. $f'(x) < 0$) in the interval (a, b), then the function f(x) is strictly decreasing in the interval (a, b).

(iii) If the derivative $f'(x)$ is everywhere equal to zero in the interval (a, b), then the function f(x) does not vary in the interval (a, b) (i.e. it is constant).

It should be borne in mind that the conditions of the theorem are sufficient, but not necessary, for a function to increase (or decrease). There are cases when a function f can increase at a point x, but the derivative $f'(x)$ is not positive there. Consider, for instance, the function $f(x) = x^3$ at $x = 0$.

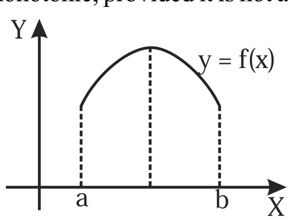
STUDY TIP

If a function is such that $f'(x) \geq 0$ for all $x \in (a, b)$ where $f'(x) = 0$ at discrete points in (a, b), then f(x) is strictly increasing in (a, b).

Note: By $f'(x) = 0$ at discrete points, we mean that the points where $f'(x)$ becomes 0 do not form an interval. That is, they are separated from each other.

A function which is increasing as well as decreasing in an interval, is said to be non-monotonic, provided it is not a constant function. If the function is differentiable, then it must change the sign of its derivative somewhere in the interval.

A graph of such a function is shown here.



Example 3. Prove that $f(x) = 2x - \cos x$ is a strictly increasing function.

Solution Given $f(x) = 2x - \cos x$

$\Rightarrow f'(x) = 2 + \cos x$

We have $f'(x) > 0$ for all x.

Hence, the function f(x) is strictly increasing for all x.

Example 4. Prove that $f(x) = -x - \cot^{-1}x$ is a strictly decreasing function.

Solution Given $f(x) = -x - \cot^{-1}x$

$$\Rightarrow f'(x) = -1 + \frac{1}{1+x^2} = -\frac{x^2}{1+x^2}$$

We have $f'(x) \leq 0$ for all x.

$f'(x) = 0$ at $x = 0$ only.

Now, $f'(x) < 0$ everywhere except at $x = 0$. Hence we can conclude that f(x) is strictly decreasing for all x.

Example 5. Is the function sin (cos x) increasing or decreasing on the interval $(\pi, 3\pi/2)$?

Solution Let $f(x) = \sin(\cos x)$

$\Rightarrow f'(x) = \cos(\cos x) \cdot (-\sin x)$

On the interval $(\pi, 3\pi/2)$, cos x lies in (–1, 0). Hence, $\cos(\cos x) > 0$.

$(-\sin x) > 0$ on the interval $(\pi, 3\pi/2)$.

Thus, we have $f'(x) > 0$ for all x in $(\pi, 3\pi/2)$.

Hence, the function f(x) is strictly increasing for all x.

Example 6. Find the behaviour of the function

$$f(x) = \begin{cases} e^x, & x < 0, \\ \ln(1+x) + 1, & x \geq 0 \end{cases} \text{ for } x \in R.$$

Solution Given $f(x) = \begin{cases} e^x, & x < 0, \\ \ln(1+x) + 1, & x \geq 0 \end{cases}$

f(x) is continuous for all x.

$$f'(x) = \begin{cases} e^x, & x < 0, \\ \dfrac{1}{1+x}, & x > 0 \end{cases}$$

Here, $f'(0^-) = 1 = f'(0^+)$

We have $f'(x) > 0$ for all $x \in R$.

Hence, we can conclude that f(x) is a strictly increasing function for all x.

Example 7. Find the behaviour of the function

$$f(x) = \begin{cases} 1, & x < 0, \\ x^3 + 1, & x \geq 0 \end{cases} \text{ for } x \in R.$$

Solution Given $f(x)=\begin{cases}1, & x<0,\\ x^3+1, & x\ge 0\end{cases}$

f(x) is continuous for all x.

$$f'(x)=\begin{cases}0, & x<0,\\ 3x^2, & x>0\end{cases}$$

Here, $f'(0^-)=0=f'(0^+)$

We have $f'(x)\ge 0$ for all $x\in R$.

But, $f'(x)=0$ in the interval $x\in(-\infty,0]$.

Hence, we can conclude that f(x) is a non-decreasing function for all x.

Example 8. Find the least value of k for which the function x^2+kx+1 is a strictly increasing function in the interval $1\le x\le 2$.

Solution Let $f(x)=x^2+kx+1$

For f(x) to be strictly increasing, $f'(x)\ge 0$ in the interval $1\le x\le 2$.

$\Rightarrow\ 2x+k\ge 0\ \Rightarrow\ k\ge -2x$

Here k must be greater than or equal to the largest value of $-2x$ found in the interval [1, 2]

i.e. $k\ge -2$

Hence, the least value of k is –2.

Example 9. For what values of b, the function $f(x)=\sin x-bx+c$ decreases strictly for all $x\in R$?

Solution Here $f(x)=\sin x-bx+c$

$\therefore\ f'(x)=\cos x-b$

f(x) will decrease for all $x\in R$ if $f'(x)\le 0$

or $\cos x-b\le 0$, i.e., $\cos x\le b$ for all $x\in R$.

$\therefore\ b\ge$ the greatest value of cos x

$b\ge 1$

Thus, $b\in[1,\infty)$.

Note that when $b=1$, $f'(x)=\cos x-1\le 0$

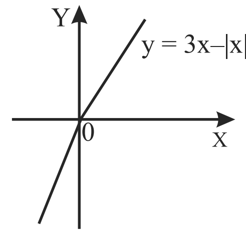
Here $f(x)=0$ at $x=2n\pi$, which are a set of discrete points, not forming an interval. Hence, f(x) decreases strictly for all $x\in R$.

Monotonicity at points where f′(x) does not exist

We have investigated the case where a function has a derivative at all points on some interval. Now what about those points at which there is no derivative ? The following examples will help in identifying the behaviour of such a function.

(i) Let $f(x)=3x-|x|$

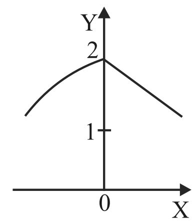
The given function has no derivative at the point x = 0, as it is a corner point.



For all $x\ne 0$, $f'(x)>0$. The function is continuous for all x with $f'(0-h)>0$ and $f'(0+h)>0$.

Hence, f(x) is a strictly increasing function for all x.

(ii) Now, consider the function $f(x)=\begin{cases}2-x^2, & x<0,\\ 2-x, & x\ge 0\end{cases}$.



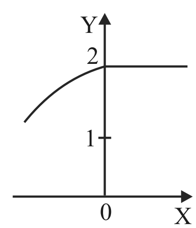
$$f'(x)=\begin{cases}-2x, & x<0,\\ -1, & x>0\end{cases}$$

Here, $f'(0-h)>0$, $f'(0^-)=0$, $f'(0^+)=-1$.

We notice that f′(x) changes sign about x = 0.

Thus, f(x) is not strictly increasing function at x = 0. In fact, the function is non-monotonous.

(iii) Now, consider the function $f(x)=\begin{cases}2-x^2, & x<0,\\ 2, & x\ge 0\end{cases}$.



$$f'(x)=\begin{cases}e^{-x}, & x<0,\\ 0, & x>0\end{cases}$$

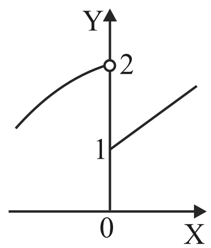
We notice that $f'(x)\ge 0$ for all $x\ne 0$.

Here, $f'(0^-)=1$, $f'(0^+)=0$, $f'(0+h)=0$.

In fact, $f'(x)=0$ for all $x>0$.

Thus, f(x) is a non-decreasing function.

(iv) Now, consider the function $f(x)=\begin{cases}2-x^2, & x<0,\\ x+1, & x\ge 0\end{cases}$.



$$f'(x)=\begin{cases}-2x, & x<0,\\ 1, & x>0\end{cases}$$

For all $x \neq 0$, $f'(x) > 0$. The function is discontinuous at $x = 0$. We can say that f(x) is a strictly increasing function for $x \in (-\infty, 0)$ and for $x \in (0, \infty)$.

We need to check the monotonicity of the function at $x = 0$ using basic definition. We can see from the figure that $f(0 - h) > f(0)$ and $f(0) < f(0 + h)$. This means that f(x) is not increasing function at $x = 0$. Hence, f(x) is not a strictly increasing function for all x.

We can now understand that in the case of continuous functions, the sign of the derivative in the neighbourhood of the point is adequate in determining the monotonicity of the function. If the derivative maintains the same sign across the point, the function is monotonous.

However, in the case of discontinuous functions, the sign of the derivative in the neighbourhood of the point is inadequate in determining the monotonicity of the function. We need to apply the basic definition of monotonicity.

Test for finding monotonicity at points where $f'(x)$ does not exist

Consider a continuous function f(x) whose derivative $f'(x)$ does not exist at $x = c$ but exists in the neighbourhood of c.

(i) If $f'(c^-) > 0$ and $f'(c^+) > 0$, then f(x) is strictly increasing at $x = c$.

(ii) If $f'(c-h)>0, f'(c^-)\geqq 0, f'(c^+)\geqq 0, f'(c+h)>0$, then f(x) is strictly increasing at $x = c$.

(iii) If $f'(c^-) < 0$ and $f'(c^+) < 0$, then f(x) is strictly decreasing at $x = c$.

(iv) If $f'(c-h)>0, f'(c^-)\leqq 0, f'(c^+)\leqq 0, f'(c+h)>0$, then f(x) is strictly increasing at $x = c$.

Here, $\geqq$ implies that either greater than or equal to holds.

Now, consider a function f(x) which is discontinuous at $x = c$ but its derivative exists in the neighbourhood of c.

(i) If $f'(c-h)>0, f(c^-)\leqq f(c), f(c)\leqq f(c^+), f'(c+h)>0$, then f(x) is strictly increasing at $x = c$.

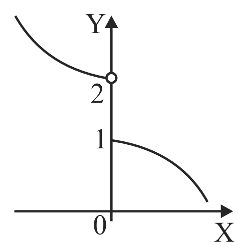
(ii) If $f'(c-h)<0, f(c^-)\geqq f(c), f(c)\geqq f(c^+), f'(c+h)<0$, then f(x) is strictly decreasing at $x = c$.

Example 10. Find the behaviour of the function

$$f(x)=\begin{cases}x^2e^{-x}+2, & x<0,\\ 1-x^2, & x\ge 0\end{cases}\text{ for } x\in R.$$

Solution Given $f(x)=\begin{cases}x^2e^{-x}+2, & x<0,\\ 1-x^2, & x\ge 0\end{cases}$

$$\Rightarrow\ f'(x)=\begin{cases}(2x-x^2)e^{-x}, & x<0,\\ -2x, & x>0\end{cases}$$



For all $x\neq 0$, $f'(x)<0$.

The function is discontinuous at $x = 0$.

Here, $f'(0-h)<0, f'(0+h)<0$,

$f(0)=1,\ f(0^-)=2>f(0)$, and $f(0)=f(0^+)$.

The value of the function is falling across the point $x=0$. Hence, f(x) is strictly decreasing at $x = 0$ and finally it is strictly decreasing for all $x \in R$.

Example 11. Let $f(x)=\begin{cases}x^3+x^2+10x, & x<0\\ x^2e^x, & x\ge 0\end{cases}$.

Investigate the behaviour of the function for $x \in R$.

Solution We have

$$f'(x)=\begin{cases}3x^2+2x+10, & x<0,\\ x(x+2)e^x, & x>0\end{cases}$$

For $x < 0$, $f'(x)$ is a quadratic expression whose discriminant is positive, with coefficient of x^2 positive. Hence it is positive. Clearly $f'(x)>0$ for $x>0$.

Thus, $f'(x) > 0$ for all $x \neq 0$.
Now consider the point $x = 0$. $f(x)$ is continuous but non-differentiable at $x = 0$.
Here, $f'(0^-) = 10 > 0, f'(0^+) = 0, f'(0+h) > 0$.
This means that f is strictly increasing at $x = 0$.
Finally, $f(x)$ is a strictly increasing function for $x \in R$.

Example 12. Show that $f(x) = \begin{cases} x^2 + 2x, & -1 \le x < 0 \\ 3x, & 0 \le x \le 1 \end{cases}$

is strictly increasing in $[-1, 1]$, but

$$g(x) = \begin{cases} x^2 + 2x, & -1 \le x < 0 \\ 3x, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}$$

is not strictly increasing in $[-1, 1]$.

Solution We have $f'(x) = \begin{cases} 2x + 2, & -1 \le x < 0 \\ 3, & 0 < x \le 1 \end{cases}$.

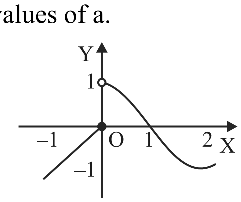
For all $x \neq 0$, $f'(x) > 0$. f is continuous at $x = 0$ and , $f'(0^-) > 0$ and $f'(0^+) > 0$.
Hence $f(x)$ is strictly increasing at $x = 0$.
At $x = -1$, $f'(-1^+) = 0$ and $f'(-1 + h) > 0$,
At $x = 1$, $f'(1^-) > 0$.
Hence $f(x)$ is strictly increasing at both the endpoints.
Finally, $f(x)$ is strictly increasing in $[-1, 1]$.

Now, $g'(x) = \begin{cases} 2x + 2, & -1 \le x < 0 \\ 3, & 0 < x < 1 \end{cases}$

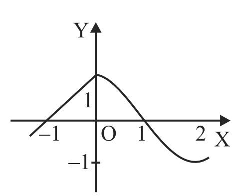
For all $x \neq 0, 1$, $g'(x) > 0$.
At $x = 0$ and -1, $g(x)$ is strictly increasing like $f(x)$.
At $x = 1$, $g(x)$ is discontinuous
where $g(1^-) = 3$ while $g(1) = 1$.
Hence, $g(x)$ is not strictly increasing at $x = 1$.
Thus, $g(x)$ is not strictly increasing in $[-1, 1]$.
However, $g(x)$ is strictly increasing in $[-1, 1)$.

Example 13. Let $f(x) = \begin{cases} \cos\dfrac{\pi x}{2}, & x > 0 \\ x + a, & x \le 0 \end{cases}$. Find the values of a if $f(x)$ is monotonous at $x = 0$.

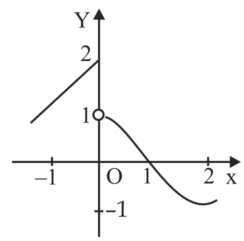
Solution We draw the graph of f with different values of a.



(a)



(b)



(c)

(a) Clearly, $f(0-h) < f(0) < f(0+h)$.
Hence $f(x)$ is strictly increasing at $x = 0$.
For this case, $f(0)$ should be less than the R.H.L. of f at $x = 0 \Rightarrow a < 1$.

(b) In this case $f(0-h) < f(0) > f(0+h)$ and hence $f(x)$ is non-monotonous at $x = 0$.

(c) Here also $f(0-h) < f(0) > f(0+h)$ and hence $f(x)$ is non-monotonous at $x = 0$.
Hence $f(x)$ is monotonous at $x = 0$ for $a < 1$.

Example 14. Find the behaviour of the function

$f(x) = \begin{cases} x, & x \le 0, \\ x\sin\dfrac{1}{x}, & x \ge 0 \end{cases}$ at $x = 0$.

Solution Clearly f is continuous at $x = 0$.

We have $f'(x) = \begin{cases} 1, & x < 0, \\ \sin\dfrac{1}{x} - \dfrac{1}{x}\cos\dfrac{1}{x}, & x > 0 \end{cases}$

Here, $f'(0^-) = 1 > 0$ but $f'(0^+)$ does not exist.
Further, $f'(0+h)$ changes sign in the right neighbourhood
since $f'(x) > 0$ at the points $x = \dfrac{1}{(2n+1)\pi}$ $(n \in N)$, and

$f'(x) < 0$ at the points $x = \dfrac{1}{2n\pi}$ $(n \in N)$,
Hence, $f(x)$ is non-monotonous at $x = 0$.

Example 15. Prove that the function

$$f(x) = \begin{cases} x + x^2\sin(2/x) & \text{for} \quad x \neq 0 \\ 0 & \text{for} \quad x = 0 \end{cases},$$

increases at the point $x = 0$ but does not increase on any interval $(-\varepsilon, \varepsilon)$, $(\varepsilon > 0$ is an arbitrary number).

Solution $f'(x) = \begin{cases} 1 + 2x\sin(2/x) - 2\cos(2/x) & \text{for} \quad x \neq 0, \\ 1 & \text{for} \quad x = 0 \end{cases}$

Since $f'(0) = 1 > 0$, it follows that the function $f(x)$ increases at the point $x = 0$.

If the function f(x) were increasing on an interval $(-\varepsilon, \varepsilon)$, then the condition $f'(x) \geq 0$ would be satisfied $\forall\, x \in (-\varepsilon, \varepsilon)$. We shall show that this is not so.
Let $x_n = 1/(n\pi)$ (n is a natural number).
It is evident that $\forall\, \varepsilon > 0$, there exists n such that $1/(n\pi) < \varepsilon$, i.e. $x_n \in (-\varepsilon, \varepsilon)$. Substituting $x = x_n = 1/(n\pi)$ into the expression for $f'(x)$ when $x \neq 0$,
we get $f'(x_n) = -1 < 0$. This proves that the function f(x) is not increasing on any interval $(-\varepsilon, \varepsilon)$.

Example 16. Let the function $g : R \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be given by $g(t) = \frac{\pi}{2} - 2\cot^{-1}(3^{-t})$. Prove that g is odd and is strictly decreasing in $(-\infty, \infty)$.

Solution We have $g(t) = \frac{\pi}{2} - 2\cot^{-1}(3^{-t})$

$$\therefore \quad g(-t) = \frac{\pi}{2} - 2\cot^{-1}(3^{t}) = \frac{\pi}{2} - 2\tan^{-1}(3^{-t})$$

$$\left(\text{As } \cot^{-1}x = \tan^{-1}\frac{1}{x},\ x > 0\right)$$

$$= \frac{\pi}{2} - 2\left(\frac{\pi}{2} - \cot^{-1}(3^{-t})\right) = -\frac{\pi}{2} + 2\cot^{-1}(3^{-t})$$

$$\left(\text{As } \cot^{-1}x + \tan^{-1}x = \frac{\pi}{2},\ \forall\ x \in R\right)$$

$$= -g(t)$$

Hence, $g(-t) = -g(t) \Rightarrow$ g is an odd function.

Also $\quad g'(t) = \dfrac{-2.3^{-t} \cdot \ln 3}{1 + (3^{-t})^2}$.

$\therefore\ g'(t) < 0,\ \forall\ t \in R$.

$\Rightarrow$ g is strictly decreasing in $(-\infty, \infty)$.

Example 17. Find the set of all values of 'a' for which $f(x) = \left(\frac{\sqrt{(a+4)}}{(1-a)} - 1\right)x^5 - 3x + \ln 5$ decreases for all x.

Solution $f(x) = \left(\frac{\sqrt{(a+4)}}{(1-a)} - 1\right)x^5 - 3x + \ln 5$

$$\therefore \quad f'(x) = 5\left(\frac{\sqrt{(a+4)}}{(1-a)} - 1\right)x^4 - 3$$

Since f(x) decreases for all x, $f'(x) \leq 0$

$$\Rightarrow \quad 5\left(\frac{\sqrt{(a+4)}}{(1-a)} - 1\right)x^4 - 3 \leq 0$$

$$\Rightarrow \quad 5\left(\frac{\sqrt{(a+4)}}{(1-a)} - 1\right) \leq \left(\frac{3}{5x^4}\right)$$

The L.H.S. should be less than or equal to the least value of R.H.S.

$$\Rightarrow \quad 5\left(\frac{\sqrt{(a+4)}}{(1-a)} - 1\right) \leq 0$$

$$\Rightarrow \quad \frac{\sqrt{(a+4)}}{(1-a)} \leq 1 \qquad \text{...(1)}$$

It is clear that $a + 4 \geq 0$

Case I : If $1 - a > 0$ i.e. $a < 1$

then $\sqrt{(a+4)} \leq (1-a)$

On squaring, we get $a + 4 \leq a^2 - 2a + 1$

$\Rightarrow\ a^2 - 3a - 3 \geq 0$

$$\Rightarrow \quad a \in \left(-\infty, \frac{3-\sqrt{21}}{2}\right] \cup \left[\frac{3+\sqrt{21}}{2}, \infty\right) \text{ but } -4 \leq a < 1$$

$$\therefore \quad a \in \left[-4, \frac{3-\sqrt{21}}{2}\right] \qquad \text{...(2)}$$

Case II : If $1 - a < 0 \Rightarrow$ i.e. $a > 1$ ...(3)

$\Rightarrow\ \sqrt{(a+4)} \geq (1-a)$ which is always true for $a > 1$ since R.H.S. is negative.

Combining (2) and (3), we get

$$a \in \left[-4, \frac{3-\sqrt{21}}{2}\right] \cup (1, \infty).$$

Example 18. Prove that the function $f(x) = \frac{\ln x}{x}$ is strictly decreasing in (e, ∞). Hence, prove that $303^{202} < 202^{303}$.

Solution We have $f(x) = \frac{\ln x}{x}$, $x > 0$.

Then $f'(x) = \frac{1 - \ln x}{x^2} < 0,\ \forall\ x > e$

$\Rightarrow$ f(x) strictly decreases in (e, ∞).

When a function is strictly decreasing, and $x_1 < x_2$ then $f(x_1) > f(x_2)$

Thus, we have $f(303) < f(202)$ as $303 > 202$.

$$\Rightarrow \quad \frac{\ln(303)}{303} < \frac{\ln(202)}{202}$$

$\Rightarrow\ 202 \ln(303) < 303 \ln(202)$

$\Rightarrow\ 303^{202} < 202^{303}$

which is the desired result.

Example 19. If $f(x) = x^3 - x^2 + 100x + 2000$, then prove that

(i) $f(1000) < f(1001)$

(ii) $f\left(\frac{1}{2000}\right) > f\left(\frac{1}{2001}\right)$

(iii) $f(x-1) > f(x-2)$

(iv) $f(2x-3) > f(x)$ for $x > 3$.

Solution $f(x) = x^3 - x^2 + 100x + 2000$

$f'(x) = 3x^2 - 2x + 100 > 0 \ \forall \ x \in R$

$\therefore$ $f(x)$ is strictly increasing $\forall \ x \in R$.

Since $1000 < 1001$ and $f(x)$ is strictly increasing $\forall$ x, we have $f(1000) < f(1001)$.

Also $\frac{1}{2000} > \frac{1}{2001}$

$\therefore \ f\left(\frac{1}{2000}\right) > f\left(\frac{1}{2001}\right)$

We have $f(x-1) > f(x-2)$ as $x - 1 > x - 2$ for $\forall$ x and $f(2x-3) > f(x)$ for such x for which $2x - 3 > x$ i.e. $x > 3$.

Example 20. Let f(x) and g(x) be two continuous function defined from $R \to R$, such that $f(x_1) > f(x_2)$ and $g(x_1) < g(x_2)$, $\forall \ x_1 > x_2$, then find the solution set of $f(g(\alpha^2 - 2\alpha)) > f(g(3\alpha - 4))$.

Solution Obviously, f is an increasing function and g is a decreasing function.

Hence $f(g(\alpha^2 - 2\alpha)) > f(g(3\alpha - 4))$

$\Rightarrow$ $g(\alpha^2 - 2\alpha) > g(3\alpha - 4)$ as f is increasing

$\Rightarrow$ $\alpha^2 - 2\alpha < 3\alpha - 4$ as g is decreasing

$\Rightarrow$ $\alpha^2 - 5\alpha + 4 < 0$

$\Rightarrow$ $\alpha \in (1, 4)$.

Example 21. Let $f(x) = 1 - x - x^3$.

Find all real values of x satisfying the inequality,

$1 - f(x) - f^3(x) > f(1 - 5x)$.

Solution $f(x) = 1 - x - x^3$

$\Rightarrow$ $f'(x) = -1 - 3x^2$ which is negative $\forall \ x \in R$

$\Rightarrow$ f is strictly decreasing.

$f[f(x)] = 1 - f(x) - f^3(x)$

$\therefore$ $f[f(x)] > f(1 - 5x)]$

Since, f(x) is strictly decreasing

$f(x_1) > f(x_2) \quad \Rightarrow x_1 < x_2$

$\therefore$ $f(x) < 1 - 5x$

$1 - x - x^3 < 1 - 5x$

$x^3 - 4x > 0$

$x(x^2 - 4) > 0$

$-$ $+$ $-$ $+$

$-2 \quad 0 \quad 2$

$\therefore$ $x \in (-2, 0) \cup (2, \infty)$.

Example 22. Prove that $(1 + x^{-1})^{1+x}$ is a strictly decreasing function for $x > 0$ with limit e as $x \to \infty$.

Solution Let $f(x) = (1 + 1/x)^{1+x}$.

Then $f'(x) = f(x).g(x)$, where $g(x) = \ln(1 + 1/x) - 1/x$.

Since f(x) is positive, we must prove $g(x) < 0$ for $x > 0$.

Now $g'(x) = 1/(x^2(x+1)) > 0$ for $x > 0$,

so g(x) is strictly increasing.

Since $g(x) \to 0$ as $x \to \infty$, we have $g(x) < 0$ for $x > 0$.

The simplest way to find $\lim\limits_{x \to \infty} f(x)$ is to write $f(x) = (1 + 1/x)(1 + 1/x)^x$.

Clearly $(1 + 1/x) \to 1$, and $(1 + 1/x)^x \to e$.

Example 23. If $f(x) = \frac{x^2}{2 - 2\cos x}$ and $g(x) = \frac{x^2}{6x - 6\sin x}$ where $0 < x < 1$, then show that f is a strictly increasing and g is a strictly decreasing function .

Solution $f'(x) = \frac{1}{2}\left[\frac{(1 - \cos x)\,2x - x^2 \sin x}{(1 - \cos x)^2}\right]$

Now consider the numerator as

$p(x) = 2(1 - \cos x) - x \sin x$

$= 4\sin^2\frac{x}{2} - 2x \sin\frac{x}{2}\cos\frac{x}{2}$

$= 2x \sin\frac{x}{2}\cos\frac{x}{2}\left[\frac{\tan\frac{x}{2}}{\frac{x}{2}} - 1\right] > 0$

$\Rightarrow f'(x) > 0 \Rightarrow$ f is strictly increasing .

Now, $g'(x) = \frac{1}{6}\left[\frac{(x - \sin x)\,2x - x^2(1 - \cos x)}{(x - \sin x)^2}\right]$

Again, consider the numerator as

$q(x) = x - 2\sin x + x\cos x$

$= 2x\cos^2\frac{x}{2} - 4\sin\frac{x}{2}\cos\frac{x}{2}$

$= 2x\cos^2\frac{x}{2}\left[1 - \frac{\tan\frac{x}{2}}{\frac{x}{2}}\right] < 0$

$\Rightarrow g'(x) < 0 \Rightarrow$ g is strictly decreasing.

Example 24. Find possible values of a such that $f(x) = e^{2x} - (a+1)e^x + 2x$ is strictly increasing for $x \in R$.

Solution $f(x) = e^{2x} - (a+1)e^x + 2x$

$f'(x) = 2e^{2x} - (a+1)e^x + 2$

Now, $2e^{2x} - (a+1)e^x + 2 \geq 0$ for all $x \in R$

$\Rightarrow \quad 2\left(e^x + \frac{1}{e^x}\right) - (a+1) \geq 0$ for all $x \in R$

$(a+1) \leq 2\left(e^x + \frac{1}{e^x}\right)$ for all $x \in R$

$\Rightarrow \quad a + 1 \leq 4 \quad \left(\because e^x + \frac{1}{e^x} \text{ has the least value } 2\right)$

$\Rightarrow \quad a \leq 3.$

***Alternative*:**

$2e^{2x} - (a+1)e^x + 2 \geq 0$ for all $x \in R$

Putting $e^x = t$ where $t \in (0, \infty)$

$2t^2 - (a+1)t + 2 \geq 0$ for all $t \in (0, \infty)$

Hence either

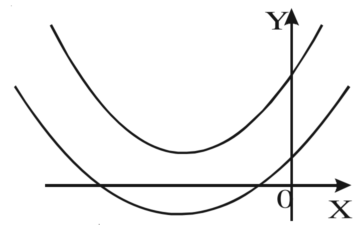
(i) $D \leq 0$

$\Rightarrow \quad (a+1)^2 - 4 \leq 0$

$\Rightarrow \quad (a+5)(a-3) \leq 0$

$\Rightarrow \quad a \in [-5, 3]$

or (ii) both roots are negative



We solve $D \geq 0, -\frac{b}{2a} < 0$ and $f(0) \geq 0$

$D \geq 0 \Rightarrow \quad a \in (-\infty, -5] \cup [3, \infty)$

$-\frac{b}{2a} < 0 \Rightarrow \frac{a+1}{4} < 0 \Rightarrow a < -1$

$f(0) \geq 0 \Rightarrow 2 \geq 0 \Rightarrow a \in R$

Hence, $a \in (-\infty, -5]$

Taking union of (i) and (ii), we get $a \in (-\infty, 3]$.

Concept Problems B

1. Find out whether each of the following statements is true:
   (i) "If a function increases at a point x_0, then it has a positive derivative at the point".
   (ii) "If the function f(x) differentiable at a point x_0 increases at that point, then $f'(x_0) > 0$".
2. Find out whether the following statement is true : "If the function f(x) differentiable on an interval X increases on that interval, then $f'(x) > 0 \ \forall \ x \in X$".
3. Let the function f(x) be defined in a neighbourhood of every point of the set X. Find out whether each of the following statements is true :
   (i) "If f(x) increases on the set X, then it increases at every point $x_0 \in X$".
   (ii) "If f(x) increases at every point $x_0 \in X$, then it increases on the set X". (Consider the function $f(x) = -1/x$).
4. Prove that if a function increases at every point of an open interval, then it increases on that interval. Will the statement remain true if we replace the interval by an arbitary set?
5. Suppose that f is an increasing function on [a, b] and that x_0 is a number in (a, b). Prove that if f is differentiable at x_0, then $f'(x_0) \geq 0$.
6. Show that if f(x) is strictly decreasing on an interval I where it is differentiable, then $f'(x) \leq 0$ for all x in I.
7. Prove that the following functions are strictly increasing.
   (i) $f(x) = \cot^{-1}x + x$
   (ii) $f(x) = \ln(1+x) - \frac{2x}{2+x}$
8. Is the function cos (sin t) increasing or decreasing on the closed interval $[-\pi/2, 0]$?
9. (i) Show that $g(x) = 1/x$ decreases on every interval in its domain.
   (ii) If the conclusion in (a) is really true, how do you explain the fact that $g(1) = 1$ is actually greater than $g(-1) = -1$?
10. Find the monotonicity of the following functions for $x \in R$.
    (i) $f(x) = \begin{cases} 2x, & x < 0, \\ 3x+5, & x \geq 0 \end{cases}$
    (ii) $f(x) = \begin{cases} 2x^3+3, & x \neq 0, \\ 4, & x = 0 \end{cases}$

11. Let $f(x)=\begin{cases} x^3, & x \le 0 \\ 2\sin 2x, & 0 < x \le a \end{cases}$.
Find the largest value of a so that f(x) is a monotonous function.

12. Find the behaviour of the function

$$f(x)=\begin{cases} 2x, & x<0 \\ 2\cos x, & x \ge 0 \end{cases} \text{ at } x=0.$$

13. Find the behaviour of the function

$$f(x)=\begin{cases} x^2, & x<0 \\ -1, & x=0 \\ 2\sin(x-\pi), & x>0 \end{cases} \text{ at } x=0.$$

14. Suppose an odd function is known to be increasing on the interval $x > 0$. What can be said of its behaviour on the interval $x < 0$?

15. Suppose an even function is known to be increasing on the interval $x < 0$. What can be said of its behaviour on the interval $x > 0$?

16. Find the value of a in order that $f(x) = \sqrt{3}\sin x - \cos x - 2ax + b$ decreases for all real values of x.

17. If $f(x)=\dfrac{2x}{1-x^2}$ strictly increasing in its domain?

18. Show that the function $y = \dfrac{x^2-1}{x}$ increases in any interval not containing the point $x = 0$

19. Find the values of k for which the function $f(x) = (k-1)x + k^2 - 3,\ x \in (-\infty, \infty)$ is
(i) strictly increasing (ii) strictly decreasing.

Practice Problems A

20. Prove that $f(x)=\dfrac{3}{2}x - \sin^2 x$ increases for $x \in R$.

21. Show that $f(x) = \dfrac{x}{\sqrt{a^2+x^2}}$ is an increasing function of x.

22. Show that $g(x)=\dfrac{d-x}{\sqrt{b^2+(d-x)^2}}$ is a decreasing function of x.

23. Prove that the function tan (cost) is decreasing on the closed interval $[0, \pi/2]$.

24. Is the function cos (sin (cos t)) increasing or decreasing on the closed interval $[\pi/2, \pi]$?

25. Prove that the function $f(x) = (1 + x)^{3/2} - \dfrac{3}{2}x - 1$ is strictly increasing on $(0, \infty)$.

26. Prove that for $a \in \left(0, \dfrac{20}{9}\right)$ the function $f(x) = x^5 + a(x^3 + x) + 1$ is invertible.

27. For which values of the constant k is the function $7x + k\sin 2x$ always increasing?

28. For what values of a is the function $f(x) = x^3 - ax$ strictly increasing for all x ?

29. If $a^2 - 3b + 15 < 0$, then show that $f(x) = x^3 + ax^2 + bx + 5\sin^2 x$ is an increasing function for all x.

30. If $f(x) = e^x(x^2 - x + 2) - (x^2 + x + 2)$, prove that when x is positive, f(x) increases as x increases.

31. If the function $f(x) = (a + 2)x^3 - 3ax^2 + 9ax - 1$ is strictly decreasing $\forall\, x \in R$, find 'a'.

32. Prove that $f(x) = \dfrac{2}{3}x^9 - x^6 + 2x^3 - 3x^2 + 6x - 1$ is strictly increasing.

33. Show that the function $f(x) = \dfrac{x}{\sqrt{1+x}} - \ell n(1 + x)$ is an increasing function when $x > -1$.

34. Find the set of all real values of μ so that the function $f(x) = (\mu + 1)x^3 + 2x^2 + 3\mu x - 7$ is
(i) strictly increasing (ii) strictly decreasing.

35. Prove that the function

$$f(x) = \begin{cases} \dfrac{1}{2}x + x^2 \sin\dfrac{1}{x} & \text{for } x \ne 0, \\ 0 & \text{for } x = 0 \end{cases}$$

is not monotonic in any interval containing the origin.

36. Show that the derivative

$$f'(x) = \frac{1}{2} + 2x\sin\frac{1}{x} - \cos\frac{1}{x},\ (x \ne 0), \text{ is equal to}$$

$\dfrac{3}{2}$ at the points $x = \dfrac{1}{(2n+1)\pi}$ $(n = 0, \pm 1, \pm 2, ...)$,

and to $-\dfrac{1}{2}$ at the points $x = \dfrac{1}{2n\pi}$, i.e. the derivative changes sign in any vicinity of the origin.

37. Suppose that f is increasing on every closed interval [a, b] provided that $2 \le a < b$. Prove that f is increasing on the unbounded open interval $(2, \infty)$.

38. Show that $y = \tan^{-1}x - x$ decreases everywhere and hence deduce that $\tan^{-1} 1 + 1 > \tan^{-1} 2$.

39. For what values of a is the function,

$$f(x) = \left(\frac{a^2-1}{3}\right)x^3 + (a-1)x^2 + 2x + 1 \text{ monotonous.}$$

6.3 CRITICAL POINT

We understand that not all functions are monotonous in their domain. In general, a function increases and decreases in different parts of its domain. Suppose a function f defined in (a, b), increases in (a, c) and then decreases in (c, b), we are now interested in knowing what must have happened at x = c and how do we get c?

To answer these questions, we should first define the term 'critical points'. We have seen in the previous section that the sign of the derivative helps in determining the behaviour of a function. These points play a crucial role in finding the sign of the derivative of a function.

Definition. A critical point of a function f is an interior point c in the domain of f such the either $f'(c) = 0$ or $f'(c)$ does not exist.

Sometimes we will want to distinguish critical numbers at which $f'(x) = 0$ from those at which f is not differentiable. We will call a point on the graph of f at which $f'(x) = 0$, a **stationary point** of f.

The stationary points of f are the x-intercepts of the graph of f'.

Note:

(i) If x = c is a critical point of the function f, then it is also a critical point of the function g(x) = f(x) + k, where k is a constant.

(ii) If x = c is a critical point of the function f, then x = c + k is a critical point of the function g(x) = f(x – k), where k is a constant.
For example, x = 0 is a critical point of $f(x) = x^2$ and x = 1 is a critical point of $g(x) = (x-1)^2$.

Remark In some texts, if $f'(c) = 0$ or $f'(c)$ does not exist, then x = c is called as critical number and (c, f(c)) is called as the critical point.

Example 1. Find the critical points of the function $f(x) = x^{3/5}(4-x)$.

Solution $f'(x) = \frac{3}{5}x^{-2/5}(4-x) + x^{3/5}(-1)$

$$= \frac{3(4-x)}{5x^{2/5}} - x^{3/5}$$

$$= \frac{3(4-x)-5x}{5x^{2/5}} = \frac{12-8x}{5x^{2/5}}.$$

Therefore, $f'(x) = 0$ if $12 - 8x = 0$, that is, $x = \frac{3}{2}$, and $f'(x)$ does not exist when x = 0.

Thus, the critical points of f(x) are 0 and $\frac{3}{2}$.

Example 2. Find the critical points for the function
$$f(x) = \frac{e^x}{x-2}$$

Solution $f'(x) = \frac{(x-2)e^x - e^x(1)}{(x-2)^2} = \frac{e^x(x-3)}{(x-2)^2}$

The derivative is not defined at x = 2, but f is not defined at 2 either, so x = 2 is not a critical point. The critical points are found by solving $f'(x) = 0$:

$$\frac{e^x(x-3)}{(x-2)^2} = 0 \Rightarrow x = 3.$$

So, x = 3 is the only critical point.

Example 3. Find the critical points of
$f(x) = (x-2)^{2/3}(2x+1)$.

Solution Given, $f(x) = (x-2)^{2/3}(2x+1)$

$$\Rightarrow \quad f'(x) = \frac{2}{3}(x-2)^{-1/3}(2x+1) + (x-2)^{2/3}\cdot 2$$

or $$f'(x) = 2\left[\frac{(2x+1)}{3(x-2)^{1/3}} + (x-2)^{2/3}\right]$$

Clearly, $f'(x)$ does not exist at x = 2, so, x = 2 is a critical point.

Other critical points are given by, $f'(x) = 0$

i.e., $$2\left[\frac{(2x+1)+3(x-2)}{(x-2)^{1/3}}\right] = 0$$

$$\Rightarrow \quad 5x - 5 = 0 \Rightarrow x = 1$$

Hence, x = 1 and x = 2 are two critical points of f(x).

Example 4. Find all possible values of the parameter b for each of which the function,
$f(x) = \sin 2x - 8(b+2)\cos x - (4b^2 + 16b + 6)x$
is strictly decreasing throughout the number line and has no critical points .

Solution We have $f'(x)$

$$= 2\cos 2x + 8(b+2)\sin x - (4b^2 + 16b + 6)$$

$$= 2(1 - 2\sin^2 x) + 8(b+2)\sin x - (4b^2 + 16b + 6)$$

$$= -4[\sin^2 x - 2(b+2)\sin x + (b^2 + 4b + 1)]$$

For f to be strictly decreasing with no critical points $f'(x) < 0 \ \forall\, x \in R$

Now, $D = 4(b+2)^2 - 4(b^2 + 4b + 1)$
$= 12$ which is always positive .

Let $\sin x = y$ where $y \in [-1, 1]$
and $g(y) = y^2 - 2(b+2)y + (b^2 + 4b + 1)$
We have to find those values of 'b' for which
$g(y) > 0$ for all $y \in [-1, 1]$
The conditions are

(i) $g(-1) > 0$ and $-\dfrac{b}{2a} < -1$, or

(ii) $g(1) > 0$ and $-\dfrac{b}{2a} > 1$

The condition (i) gives
$1 + 2(b+2) + b^2 + 4b + 1 > 0$
$b^2 + 6b + 6 > 0$ ...(1)

and $\dfrac{2(b+2)}{2} < -1$ or $b < -3$ ...(2)

(1) and (2) $\Rightarrow b < -(3 + \sqrt{3})$

Similarly, the condition (ii) gives $b > \sqrt{3} - 1$

Hence, $b \in (-\infty, -(3+\sqrt{3})) \cup (\sqrt{3} - 1, \infty)$.

Practice Problems B

1. Prove that $f(x) = \dfrac{x^5}{e^x - 1}$ has two stationary points.
2. Find the stationary points of
   $f(x) = \dfrac{5x^2 - 18x + 45}{x^2 - 9}$.
3. Find the critical points of the function :
   (i) $f(t) = 3t^4 + 4t^3 - 6t^2$
   (ii) $f(x) = x^{4/5}(x-4)^2$
   (iii) $f(\theta) = 2\cos\theta + \sin^2\theta$
4. Find the critical points of the function :
   (i) $y = x + \cos^{-1} x + 1$
   (ii) $y = x\tan^{-1} x$
   (iii) $y = e^{|x|} - 2x + 1$
5. Define f(x) to be the distance from x to the nearest integer. What are the critical points of f ?
6. Find the critical points of the function :
   (i) $y = \sqrt{x^2 - 6x + 15}$ (ii) $y = (x+2)\sqrt{x-1}$.
7. Find the critical points of the function :
   (i) $f(x) = e^3 - \sqrt{4x^2 - 12x + 9} - 4\sin^2 \dfrac{x}{2}$
   (ii) $f(x) = \sin^2 3x + 3\sqrt{x^2 - 4x + 4} + \cos 1$
8. Find the critical points of the function :
   (i) $y = 3\sin x + 2(x-1)$
   (ii) $y = \cos 2x + ax - \sqrt{3}$
9. Find the critical points of the function $y = 2\sin^2 \dfrac{x}{6} + \sin \dfrac{x}{3} - \dfrac{x}{3}$. whose coordinates satisfy the inequality $x^2 - 10 < -19.5x$.
10. Find all the values of a for which the function f(x) does not posses critical points where
    $f(x) = (4a-3)(x + \ln 5) + 2(a-7)\cot \dfrac{x}{2} \sin^2 \dfrac{x}{2}$.

6.4 INTERVALS OF MONOTONICITY

We now wish to find the various intervals in which a non-monotonous function f increases or decreases. For this purpose we first find the domain of the function f. Suppose that the function is defined in [a, b]. We then find the critical points, say $x_1, x_2, ..., x_{n-1}, x_n$.

Next we split the whole interval [a, b], by means of the points x_i, into the subintervals $(a, x_1), (x_1, x_2), ..., (x_{n-1}, x_n), (x_n, b)$. In each subinterval, the derivative has constant sign. We assume that the derivative is continuous in each subinterval, and hence, if it changed the sign, $f'(x)$ would turn into zero at some new point, which is impossible since all such points have already been found as critical points (they serve as the end points of the subintervals).

Thus, these subintervals are the intervals of monotonicity of the function.

Now we must determine the sign of the derivative in each subinterval. The sign of the derivative in each subinterval can be determined by computing the value of the function $f'(x)$ at an arbitrary point of every subinterval.

If the derivative is represented as a product of a number of factors it is sufficient to determine the signs of these factors without computing their values since these signs specify the sign of the derivative. The sign of the derivative specifies the character of variation of the function in each interval of monotonicity, that is its increase or decrease.

For example, take the function
$y = 2/3x^3 - 2x^2 - 6x + 3$.
$y' = 2x^2 - 4x - 6$

Solving $y' = 0$ we get the critical points $x = -1$ and 3.

Note that y is a differentiable function for all x.

To check that in the interval $-1 < x < 3$, whether it decreases, it is sufficient to verify that its derivative $y' = 2x^2 - 4x - 6$ is negative for $-1 < x < 3$.

The latter does in fact take place since

$y' = 2(x + 1)(x - 3)$, and the factor $(x + 1)$ is positive for all the values of x in this interval while the factor $(x - 3)$ is negative.

CAUTION

It was noticed earlier that a function may be strictly monotonous even when its derivative became zero at several discrete points or the derivative didnot exist.

This implies that all critical points may not be instrumental in changing the monotonic behaviour of the function. This means that $f'(x)$ need not change sign at each critical point.

For instance, the function

$$f(x) = (x^2 - 2x + 2)e^x \text{ whose } f'(x) = x^2e^x$$

has a critical point $x = 0$, but the function does change its increasing behaviour at $x = 0$ since its derivative maintains positive sign across the point $x = 0$. Here, $f(x)$ is strictly increasing for $x \in (-\infty, \infty)$.

Steps for finding intervals of monotonicity

Let us now formulate the rule for finding the intervals of monotonicity of a function :

1. Compute the derivative $f'(x)$ of a given function $f(x)$, and then find the points at which $f'(x)$ equals zero or does not exist at all. These points are the critical points for the function $f(x)$.
2. Using the critical points, separate the domain of definition of the function $f(x)$ into several intervals on each of which the derivative $f'(x)$ retains its sign. These intervals will be the intervals of monotonicity.
3. Investigate the sign of $f'(x)$ on each of the found intervals. If on a certain interval $f'(x) > 0$, then the function $f(x)$ increases on this interval, and if $f'(x) < 0$, then $f(x)$ decreases on this interval.

Example 1. Test the function $f(x) = \frac{1}{5}x^5 - \frac{1}{3}x^2$ for increase or decrease.

Solution First, we find the derivative : $f'(x) = x^4 - x^2$

Next, we determine the critical points:

$f'(x)$ exist for all x and $f'(x) = 0$ when $x^4 - x^2 = 0$, we find the points $x_1 = -1$, $x_2 = 0$, $x_3 = 1$, at which the derivative $f'(x)$ vanishes.

Since $f'(x)$ can change sign only when passing through points at which it vanishes or becomes discontinuous (in the given case, $f'(x)$ has no discontinuities), the derivative in each of the intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$ and $(1, \infty)$ retains its sign; for this reason, the function under investigation is monotonic in each of these intervals.

To determine in which of the indicated intervals the function increases and in which it decreases, one has to determine the sign of the derivative in each of the intervals.

To determine what the sign of $f'(x)$ is in the interval $(-\infty, -1)$, it is sufficient to determine the sign of $f'(x)$ at some point of the interval; for example, taking $x = -2$, we get $f'(-2) = 12 > 0$; hence, $f'(x) > 0$ in the interval $(-\infty, -1)$ and the function in this interval increases.

Similarly, we find that $f'(x) < 0$ in the interval $(-1, 0)$ (as a check, we can take $x = -\frac{1}{2}$), $f'(x) < 0$ in the interval $(0, 1)$ (here, we can use $x = 1/2$ and $f'(x) > 0$ in the interval $(1, \infty)$.

Thus, the function increases in the interval $(-\infty, -1)$, decreases in the interval $(-1, 1)$ and again increases in the interval $(1, \infty)$.

Example 2. Determine where the function $f(x) = x^3 - 3x^2 - 9x + 1$ is strictly increasing and where it is strictly decreasing.

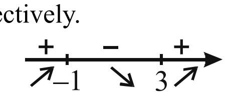
Solution First, we find the derivative :

$$f'(x) = 3x^2 - 6x - 9 = 3(x + 1)(x - 3)$$

Next, we determine the critical points:

$f'(x)$ exist for all x and $f'(x) = 0$ at $x = -1$ and $x = 3$.

These critical points divide the x-axis into three parts, and we select a typical number from each of these intervals. For example, we select -2, 0 and 4, and evaluate the derivative at these numbers, and mark each interval as increasing ($\uparrow$) or decreasing($\downarrow$), according to whether the derivative is positive or negative, respectively.



Thus, the function increases in the intervals $(-\infty, -1)$ and $(3, \infty)$ and decreases in the interval $(-1, 1)$.

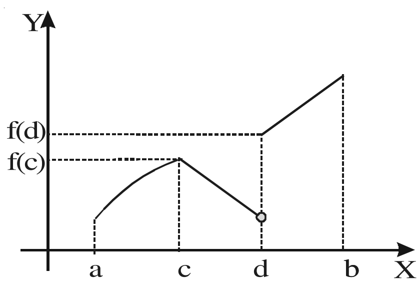
CAUTION

While writing the intervals of increase or decrease it is not advisable to use the union symbol '$\cup$', unless due care has been taken. Suppose that in the above example we write the intervals of increase as

$(-\infty, -1) \cup (3, \infty)$, then by definition it means that if $x_1, x_2 \in (-\infty, -1) \cup (3, \infty)$ where $x_1 < x_2$ then $f(x_1) < f(x_2)$ for all such x_1, x_2. This is not true since x_1 can belong to $(-\infty, -1)$ and x_2 can belong to $(3, \infty)$ and we have not checked that the maximum value of f(x) obtained in $(-\infty, -1]$ is whether less than or equal to the minimum value of f(x) obtained in $[3, \infty)$. In this function it is surely not so.

In case of discontinuous functions there is a chance for this to happen.

For instance, see the function y = f(x) graphed below :



Here, the function increases in the intervals (a, c), (d, b) and we may proceed to write that it increases in (a, c) $\cup$ (d, b) because we in this function we have $f(c) \le f(d)$.

Example 3. Find whether the function $f(x) = xe^{-3x}$, increases or decreases.

Solution We find the derivative

$$f'(x) = e^{-3x} - 3xe^{-3x} = e^{-3x}(1 - 3x).$$

The derivative f′(x) exists everywhere and vanishes at the point 1/3. The point x = 1/3 divides the number line into two intervals, $(-\infty, 1/3)$ and $(1/3, \infty)$.

Since the function e^{-3x} is always positive, the sign the derivative is decided by the second factor. Consequently $f'(x) > 0$ on the interval $(-\infty, 1/3)$ and $f'(x) < 0$ on the interval $(1/3, \infty)$.

Hence, the function f(x) increases on the interval $(-\infty, 1/3)$ and decreases on the interval $(1/3, \infty)$.

Example 4. Find the intervals of monotonicity of $f(x) = (2^x - 1)(2^x - 2)^2$.

Solution $f(x) = (2^x - 1)(2^x - 2)^2$

$$\Rightarrow f'(x) = 2^x\log2(2^x - 2)^2 + 2(2^x - 2)\log2(2^x - 1)$$
$$= 2^x\log2(2^x - 2)[(2^x - 2) + 2(2^x - 1)]$$
$$= 2^x\log2(2^x - 2)[3.2^x - 4]$$

$f'(x) = 0 \Rightarrow 2^x - 2 = 0$ or, $3.2^x - 4$

$\Rightarrow x = 1$ $\Rightarrow x = \log_2(4/3)$

Sign scheme of f′(x)

+ $\log_2(4/3)$ – 1 +

Thus, f(x) is increasing in $(-\infty, \log_2(4/3))$ and $(1, \infty)$ and decreasing in $(\log_2(4/3), 1)$.

Example 5. Find the intervals of monotonicity of the following functions :

(i) $f(x) = 2x^2 - \ln|x|$ (ii) $f(x) = \dfrac{x^3}{x^4 + 27}$.

Solution (i) We have $f(x) = 2x^2 - \ln|x|$

and $f'(x) = 4x - \dfrac{1}{x} = \dfrac{4\left(x + \frac{1}{x}\right)\left(x - \frac{1}{2}\right)}{x}$

Now, from the sign scheme for f′(x), we have

– $-1/2$ + 0 – $1/2$ +

$\Rightarrow$ f(x) strictly decreases in $(-\infty, -1/2)$
strictly increases in $(-1/2, 0)$
strictly decreases in $(0, 1/2)$
strictly increases in $(-1/2, \infty)$

Finally, f(x) increases in $\left(-\dfrac{1}{2}, 0\right), \left(\dfrac{1}{2}, \infty\right)$

and decreases in $\left(-\infty, -\dfrac{1}{2}\right), \left(0, \dfrac{1}{2}\right)$.

(ii) We have $f(x) = \dfrac{x^3}{x^4 + 27}$ and

$$f'(x) = \frac{(x^4 + 27)(3x^2) - x^3(4x^3)}{(x^4 + 27)^2} = \frac{-x^2(x^4 - 81)}{(x^4 + 27)^2}$$
$$= \frac{x^2(x^2 + 9)(x + 3)(x - 3)}{(x^4 + 27)^2}$$

Now, from the sign scheme for f′(x), we have

– -3 + 3 –

$\Rightarrow$ f(x) strictly decreases in $(-\infty, -3)$
strictly increases in $(-3, 3)$
strictly decreases in $(3, \infty)$.

Finally, f(x) increases in $(-3, 3)$
and decreases in $(-\infty, -3)$, $(3, \infty)$.

Example 6. Determine the intervals of increase and decrease of the function $y = \dfrac{1}{x+2}$.

Solution Here, $x = -2$ is a discontinuity of the function and $y' = \dfrac{1}{(x+2)^2} < 0$ for $x \neq -2$.

Hence, the function y decreases in the intervals $-\infty < x < -2$ and $-2 < x < \infty$.

Note that the function decreases in two separate intervals and we cannot say that the function is decreasing in its domain, since $f(-2^-) = -\infty$ and $f(-2^+) = \infty$. In fact the values of the function on the left of $x = -2$ are smaller than those on the right of -2.

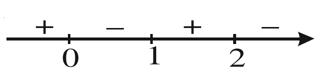
Example 7. Find the intervals of monotonicity of the function $f(x) = \dfrac{|x-1|}{x^2}$.

Solution We have $f(x) = \dfrac{1-x}{x^2}, x < 1$

$$= \frac{1-x}{x^2}, x \geq 1$$

and $f'(x) = \dfrac{-2}{x^3} + \dfrac{1}{x^2} = \dfrac{x-2}{x^3}, x < 1$

$$= \frac{2-x}{x^3},\ x > 1$$



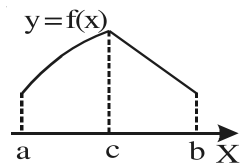
Now, from the sign scheme for $f'(x)$, we see that f(x) strictly increases in $(-\infty, 0)$, $(1, 2)$ and strictly decreases in $(0, 1)$, $(2, \infty)$.

Remark It is a convention that we write the intervals of monotonicity using open intervals but, ideally the use of closed intervals is more informative, particularly in discontinuous functions. In the case of continuous functions defined on a close interval, the open intervals of monotonicity can be easily replaced by closed intervals. However, in the case of discontinuous functions due care must be taken in using closed brackets.

For instance, consider the following functions. Using open brackets we would write in each of the functions that f(x) increases in the interval (a, c) and decreases in the interval (c, b).

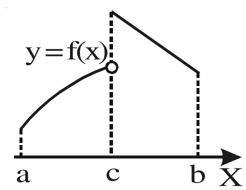
Now, we write the intervals of monotonicity using closed brackets.

(i)



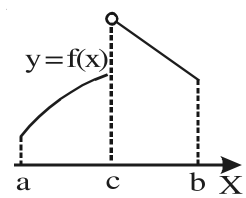
f(x) increases in the interval [a, c] and decreases in the interval [c, b].

(ii)



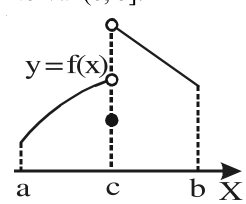
f(x) increases in the interval [a, c] and decreases in the interval [c, b].

(iii)



f(x) increases in the interval [a, c] and decreases in the interval (c, b].

(iv)



f(x) increases in the interval [a, c) and decreases in the interval (c, b].

Example 8. Find the intervals of increase of the function $f(x) = x^2\, e^{-x^2/a^2}, a > 0$

Solution $f'(x) = 2x\, e^{-x^2/a^2} + x^2 e^{-x^2/a^2} \times \dfrac{-2x}{a^2}$

$$= 2x\, e^{-x^2/a^2}\left(1 - \frac{x^2}{a^2}\right) = \frac{2e^{-x^2/a^2}}{a^2} x(a^2 - x^2)$$

$$= \text{positive no. } [-1\,(x+a)\,(x-0)\,(x-a)]$$

Sign scheme of $f'(x)$

$$\begin{array}{ccccccc} + & & - & & + & & - \\ & -a & & 0 & & a & \end{array}$$

$f(x)$ is increasing in $(-\infty, -a]$ and $[0, a]$.

Example 9. Find the intervals of monotonicity of the function $f(x) = x + \dfrac{4}{x^2}$.

Solution The function f is undefined at $x = 0$, but continuous elsewhere. Evidently $f'(x) = 1 - \dfrac{8}{x^3}$, and $f'(x) = 0$ if and only if, $x = 2$. Thus, 2 is the only critical point. Therefore, f is strictly monotonic in each of the intervals $(-\infty, 0), (0, 2], [2, \infty]$.

Since $f'(-1) > 0, f'(1) < 0, f'(3) > 0$, we have :

f is strictly increasing in $(-\infty, 0)$, strictly decreasing in $(0, 2]$, and strictly increasing in $[2, \infty)$.

Note:

(i) If f is increasing on the intervals $(a, c]$ and $[c, b)$, then f is increasing on (a, b).

(ii) If f is decreasing on the intervals $(a, c]$ and $[c, b)$, then f is decreasing on (a, b).

Remark Suppose that the function is defined in (a, b). If it increases in two consecutive intervals, say (a, c) and (c, b) then can we always write that it increases in (a, b) ? The answer is no. This can be surely done if the function is continuous at $x = c$. However, if it is discontinuous at $x = c$, then we can join the intervals only when $f(c^-) \le f(c) \le f(c^+)$.

Similarly, if the function decreases in two consecutive intervals then we can join the intervals only when $f(c^-) \ge f(c) \ge f(c^+)$.

For example, consider the function

$$f(x) = x^5 - 5x^4 + 5x^3 + 1.$$

$$f'(x) = 5x^2(x-1)(x-3)$$

The critical points are $x = 0$, 1 and 3.

Sign scheme of $f'(x)$:

$$\begin{array}{ccccccc} + & & + & & - & & + \\ & 0 & & 1 & & 3 & \end{array}$$

The function increases in $(-\infty, 0)$, $(0, 1)$ and $(3, \infty)$. Since, f is continuous at $x = 0$, we may prefer to write that f increases in $(-\infty, 1)$ and $(3, \infty)$.

Example 10. Show that $f(x) = \begin{cases} x^2 + 2x, & -1 < x < 0 \\ 3x + 1, & 0 \le x < 1 \end{cases}$ is strictly increasing in $(-1, 1)$, but

$g(x) = \begin{cases} x^2 + 2x, & -1 < x < 0 \\ 3x - 1, & 0 \le x < 1 \end{cases}$ is not strictly increasing in $(-1, 1)$.

Solution We have $f'(x) = \begin{cases} 2x + 2, & -1 < x < 0 \\ 3, & 0 < x \le 1 \end{cases}$

For all $x \ne 0$, $f'(x) > 0$.

Thus, $f(x)$ is increasing in $(-1, 0)$ and $(0, 1)$.

At $x = 0$, f is discontinuous,

however $f(0^-) = 0$, $f(0) = 1$ and $f(0^+) = 1$.

Because of the non-decreasing order of these quantities, we can proceed to say that $f(x)$ is increasing in $(-1, 1)$.

Quite similar to f, $g(x)$ is decreasing in $(-1, 0)$ and $(0, 1)$.

At $x = 0$, g is also discontinuous.

But $g(0^-) = 0$, $g(0) = 1$ and $g(0^+) = -1$.

Since, the non-decreasing order of these quantities, is not maintained, $g(x)$ cannot be said to increase in $(-1, 1)$.

Example 11. Find the intervals of increase and decrease of the function $f(x) = \dfrac{x}{\ln x}$.

Solution The function $f(x) = \dfrac{x}{\ln x}$ is defined for $x \in (0, 1) \cup (1, \infty)$.

Now, $f'(x) = \dfrac{\ln x - 1}{(\ln x)^2}$.

Let $f'(x) = 0 \Rightarrow \ln x - 1 = 0 \Rightarrow x = e$

If follows that $f'(x) > 0$ if $x > e$ and $f'(x) < 0$ if $x < e$.

Consequently, $f(x)$ decreases on the intervals $(0, 1)$ and $(1, e)$ and increases on the interval (e, ∞).

Note that here we cannot join the intervals of decrease $(0, 1)$ and $(1, e)$ because f is undefined at $x = 1$.

Example 12. A function $f(x)$ is given by the equation, $x^2 f'(x) + 2x f(x) - x + 1 = 0$ $(x \ne 0)$. If $f(1) = 0$, then find the intervals of monotonicity of f.

Solution We have $x^2 f'(x) + 2x f(x) - x + 1 = 0$

$$\Rightarrow \frac{d}{dx}[x^2 y] = x - 1 \text{ where } y = f(x)$$

$$\Rightarrow x^2 y = \int (x-1)\,dx \Rightarrow x^2 y = \frac{x^2}{2} - x + c$$

Now $f(1) = 0 \Rightarrow c = \dfrac{1}{2}$.

This gives $y = \dfrac{1}{2} - \dfrac{1}{x} + \dfrac{1}{2x^2}$.

$$\frac{dy}{dx}=\frac{1}{x^2}-\frac{1}{x^3}=\frac{x-1}{x^3}$$

$+$ 0 $-$ 1 $+$

Considering the sign of $\frac{dy}{dx}$, we find that

f(x) increases in $(-\infty, 0)$ and $(1, \infty)$ while it decreases in (0, 1).

Example 13. Let $f'(\sin x) < 0$ and $f''(\sin x) > 0$, $\forall x \in \left(0, \frac{\pi}{2}\right)$ and $g(x) = f(\sin x) + f(\cos x)$, then find the intervals in which g(x) is increasing and decreasing.

Solution Here,

$$f'(\sin x) < 0 \text{ and } f''(\sin x) > 0, \forall x \in \left(0, \frac{\pi}{2}\right) \quad ...(1)$$

and $g(x) = f(\sin x) + f(\cos x)$

$\Rightarrow$ $g'(x) = f'(\sin x)\cdot\cos x + f'(\cos x)(-\sin x)$

$\Rightarrow$ $g''(x) = \{-f'(\sin x)\cdot\sin x + f''(\sin x)\cos^2 x\}$
$- \{f'(\cos x)\cdot\cos x - f''(\cos x)\sin^2 x\}$ ...(2)

From (1), we have $f'(\cos x) < 0$ and $f''(\cos x) > 0$

$$\forall x \in \left(0, \frac{\pi}{2}\right) \quad ...(3)$$

$\therefore$ Using (1), (2) and (3), we have,

$$g''(x) = \underbrace{\{-f'(\sin x)\cdot\cos x\}}_{\text{positive}} + \underbrace{\{f''(\sin x)\cdot\cos^2 x\}}_{\text{positive}} + \underbrace{\{f''(\cos x)\cdot\sin^2 x\}}_{\text{positive}} + \underbrace{\{-f'(\cos x)\cdot\cos x\}}_{\text{positive}}$$

$\Rightarrow$ $g''(x) > 0\ \forall x \in \left(0, \frac{\pi}{2}\right)$ ...(4)

$\Rightarrow$ $g'(x)$ is strictly increasing in $\left(0, \frac{\pi}{2}\right)$.

Now putting $g'(x) = 0$, we have

$f'(\sin x)\cdot\cos x - f'(\cos x)\sin x = 0$

$\Rightarrow$ $x = \frac{\pi}{4}$ [by trial and error]

Since $g'(x)$ is increasing in $\left(0, \frac{\pi}{2}\right)$, it has atmost one root. Hence $x = \frac{\pi}{4}$ is the only critical point of g(x).

Also $g'(x) < 0$, when $x \in \left(0, \frac{\pi}{4}\right)$

$g'(x) > 0$, when $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

$\therefore$ g(x) is decreasing when $x \in \left(0, \frac{\pi}{4}\right)$

g(x) is increasing when $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

Example 14. Find the intervals of increase of g(x), where $g(x) = 2f\left(\frac{x^2}{2}\right) + f(6 - x^2)\ \forall x \in R$, given that $f''(x) > 0\ \forall x \in R$.

Solution $g(x) = 2f\left(\frac{x^2}{2}\right) + f(6 - x^2)$

$\Rightarrow$ $g'(x) = 2xf'\left(\frac{x^2}{2}\right) - 2xf'(6 - x^2)$

$$= 2x\left\{f'\left(\frac{x^2}{2}\right) - f'(6 - x^2)\right\} \quad ...(1)$$

But given that $f''(x) > 0 \Rightarrow f'(x)$ is increasing for all $x \in R$.

Case I : Let $\frac{x^2}{2} > (6 - x^2) \Rightarrow x^2 > 4$

$\Rightarrow$ $x \in (-\infty, -2) \cup (2, \infty)$

$\therefore$ $f'\left(\frac{x^2}{2}\right) > f'(6 - x^2)$

$\Rightarrow$ $f'\left(\frac{x^2}{2}\right) - f'(6 - x^2) > 0$ for $x \in (-\infty, -2)$ and $(2, \infty)$...(2)

From (1) and (2), $g'(x) > 0$ for $x \in (2, \infty)$

and $g'(x) < 0$ for $x \in (-\infty, -2)$.

Case II : Let $\frac{x^2}{2} < (6 - x^2) \Rightarrow x^2 < 4 \Rightarrow x \in (-2, 2)$

$\therefore$ $f'\left(\frac{x^2}{2}\right) < f'(6 - x^2)$

$\Rightarrow$ $f'\left(\frac{x^2}{2}\right) - f'(6 - x^2) < 0$ for $x \in (-2, 2)$ ...(3)

From (1) and (3), $g'(x) < 0$ for $x \in (0, 2)$.

and $g'(x) > 0$ for $x \in (-2, 0)$

Combining both cases, g(x) is increasing in

$x \in (-2, 0)$ and $(2, \infty)$.

Application of monotonicity in isolation of roots

Suppose that

(i) f is continuous on [a, b] and differentiable on (a, b),

(ii) f(a) and f(b) have opposite signs,

(iii) $f'(x) > 0$ on (a, b) or $f'(x) < 0$ on (a, b), then f has exactly one root between a and b.

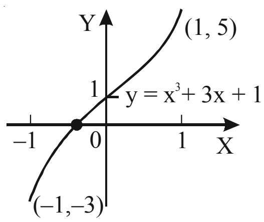
It cannot have more than one root because it is either increasing on [a, b] or decreasing on [a, b]. Yet it has atleast one root, by the Intermediate Value Theorem.

For example, $f(x) = x^3 + 3x + 1$ has exactly one zero on $[-1, 1]$ because

(i) f is differentiable on $[-1, 1]$,

(ii) $f(-1) = -3$ and $f(1) = 5$ have opposite signs, and

(iii) $f'(x) = 3x^2 + 3 > 0$ for all x in $[-1, 1]$.



Consider another example. Let us take the equation

$$f(x) = x^3 + 1.1\,x^2 + 0.9x - 1.4 = 0$$

Since $f'(x) = 3x^2 + 2.2x + 0.9 > 0$ for all the values of x, the function f(x) is strictly increasing, and hence its graph cuts the x-axis only once.

Besides, $f(0) = -1.4$ and $f(1) = 1.6$, which means that there is a single real root located within the interval [0, 1].

Let us compute $f(0.5) = -0.55$ and then $f(0.7) = 0.112$. This shows that [0.5, 0.7] is a reduced interval of isolation of the sought-for root.

Example 15. Show that the equation $x^5 - 3x - 1 = 0$ has a unique root in [1, 2].

Solution Consider the function

$$f(x) = x^5 - 3x - 1, x \in [1, 2]$$

and $f'(x) = 5x^4 - 3 > 0\ \forall\ x \in (1, 2)$

$\Rightarrow$ f(x) is strictly increasing in (1, 2).

Also, we have

$$f(1) = 1 - 3 - 1 = -3$$

and $f(2) = 32 - 6 - 1 = 25$

Hence $y = f(x)$ will cut the x-axis exactly once in [1, 2] i.e. f(x) will have a unique root in [1, 2].

Example 16. Find the values of a, if the equation

$$x - \sin x = a \text{ has a unique root in } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Solution Consider the function

$$f(x) = x - \sin x, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Then $f'(x) = 1 - \cos x = 2\sin^2\left(\frac{x}{2}\right) > 0$

$$\forall\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

$\Rightarrow$ f(x) is strictly increasing in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Also, we have $f\left(\frac{-\pi}{2}\right) = -\frac{\pi}{2} + 1 - a$

and $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 1 - a$.

The curve $y = f(x)$ will cuts the x-axis exactly once, if $f\left(\frac{-\pi}{2}\right)$ is negative or zero and $f\left(\frac{\pi}{2}\right)$ is positive or zero.

i.e. $-\frac{\pi}{2} + 1 - a \le 0$ and $\frac{\pi}{2} - 1 - a \ge 0$

i.e. $a \ge -\frac{\pi}{2} + 1$ and $a \le \frac{\pi}{2} - 1$

Hence, we have $a \in \left[1 - \frac{\pi}{2}, \frac{\pi}{2} - 1\right]$.

Example 17. Let $f(x) = x^3 + 2x^2 + x + 5$. Show that f(x) has only one real root α such that $[\alpha] = -3$.

Solution We have $f(x) = x^3 + 2x^2 + x + 5, x \in R$

and $f'(x) = 3x^2 + 4x + 1 = (x + 1)(3x + 1), x \in R$

Sign scheme of $f'(x)$

$$\begin{array}{ccccc} + & & - & & + \\ & -1 & & -1/3 & \end{array}$$

f(x) strictly increases in $(-\infty, -1)$

strictly decreases in $(-1, -1/3)$

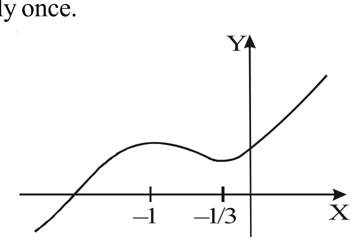
strictly increases in $(-1/3, \infty)$

Also, we have

$$f(-1) = -1 + 2 - 1 + 5 = 5$$

and $f\left(\frac{-1}{3}\right) = \frac{-1}{27} + \frac{2}{9} - \frac{1}{3} + 5 = 5 - \frac{4}{27} = 4.85$

The graph of f(x) (see figure) shows that f(x) cuts the x-axis only once.



Now, we have $f(-3) = -27 + 12 - 3 + 5 = -13$

and $f(-2) = -8 + 8 - 2 + 5 = 3$,

which are of opposite signs. This proves that the curve cuts the x-axis somewhere between –2 and –3.

$\Rightarrow f(x) = 0$ has only one real root α lying between –2 and –3. Hence, $[\alpha] = -3$.

Concept Problems C

1. Find the intervals of monotonicity of the following functions :

(i) $f(x) = ax^2 + bx + c$ $(a > 0)$,

(ii) $f(x) = x^3 + 3x^2 + 3x$,

(iii) $f(x) = \dfrac{2x}{1+x^2}$

(iv) $f(x) = x + \sin x$,

(v) $f(x) = x + 2\sin x$

(vi) $f(x) = \sin(\pi/x)$,

(vii) $f(x) = x^2 2^{-x}$

(viii) $f(x) = x^n e^{-x}$ $(n > 0, x \geq 0)$.

2. Determine the intervals of monotonicity of the following functions :

(i) $f(x) = 3x^4 - 6x^2 + 4$ (ii) $f(x) = \dfrac{x}{x^2+1}$

(iii) $f(x) = \dfrac{x^3}{1-x}$

3. Find the intervals of the increase of the function $y = \dfrac{x+2}{x^2-1}$.

4. Find the intervals of decrease of the function $y = \sqrt{2x^2 - x + 1}$.

5. Show that the equation $xe^x = 2$ has only one positive root found in the interval (0, 1).

6. Show that the function $f(x) = xe^x - 2$ increases and has opposite signs at the endpoints of the interval (0, 1).

7. (i) Show that $g(t) = \sin^2 t - 3t$ decreases on every interval in its domain.

(ii) How many solutions does the equation $\sin^2 t - 3t = 5$ have? Give reasons for your answer

8. Show that the equation $x^4 + 2x^2 - 2 = 0$ has exactly one solution on [0, 1].

Practice Problems C

9. Find the intervals of monotonicity of the given functions.

(i) $y = (x-2)^5 (2x+1)^4$

(ii) $y = x^2 e^{-x}$

(iii) $y = \dfrac{x}{\ln x}$

(iv) $y = x - 2\sin x$ $(0 \leq x \leq 2\pi)$

10. Find the intervals of monotonicity of the following functions (make use of closed bracket wherever possible):

(i) $f(x) = -x^3 + 6x^2 - 9x - 2$

(ii) $f(x) = x + \dfrac{1}{x+1}$

(iii) $f(x) = x \cdot e^{x-x^2}$

(iv) $f(x) = x - \cos x$

11. Find the intervals of increase of the function

(i) $y = |x| - \cos 2x$

(ii) $y = \begin{cases} \sin\dfrac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$

12. Determine the intervals of increase and decrease of the function $y = 2e^{x^2-4x}$.

13. Determine the intervals in which the function $f(x) = x^2 \ln 27 - 6x \ln 27 + (3x^2 - 18x + 24) \times \ln(x^2 - 6x + 8)$ is strictly decreasing.

14. Find the critical points of the function $f(x) = 4x^3 - 6x^2 \cos 2a + 3x \sin 2a \sin 6a + \sqrt{\ln(2a - a^2)}$.

Does f(x) decrease or increase at the point $x = 1/2$?

15. Show that $f(x) = x^7 + x^5 + x^3 + x + 1$ has precisely one real zero. How can this result be generalized to polynomials without even powers ?

16. Show that the equation $x^2 = x \sin x + \cos x$ holds for exactly two real values of x.

17. Show that the equation $e^{ax} = bx$, where a and b are positive, has two real roots, one, or none, according as $b > ae$, $b = ae$, or $b < ae$.

18. Show that the equation $e^x = 1 + x$ has no real root except $x = 0$, and that $e^x = 1 + x + x^2/2$ has one real root.

19. Show that the equation $3\tan x + x^2 = 2$ has exactly one solution in the interval $[0, \pi/4]$.

20. Consider the function $f(x) = x^3 + ax^2 + c$. Show that if $a < 0$ and $c > 0$, then f has exactly one negative root.

6.5 MONOTONICITY IN PARAMETRIC FUNCTIONS

Let the function y = f(x) be given by

$x = \phi(t)$

$y = \psi(t)$

One way is to eliminate t to find the function y in terms of x and then proceed as usual.

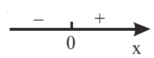
Another way is to continue with t. We first find the critical point in terms of t and then investigate the sign of $\frac{dy}{dx} = \frac{\psi'(t)}{\phi'(t)}$.

Example 1. Find the intervals of monotonicity of the function y = f(x) given by $x = \ln t$, $y = (t-1)^2$.

Solution

Method 1 : We have $t = e^x \Rightarrow y = (e^x - 1)^2$

$$\frac{dy}{dx} = 2(e^x - 1)\,e^x$$



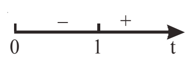
f(x) is increasing for $x \in (0, \infty)$ and it is decreasing for $x \in (-\infty, 0)$.

Example 2. $x = \ln t$, $y = (t-1)^2$, are defined for $t > 0$.

$$\frac{dy}{dx} = \frac{2(t-1)}{1/t} = 2t\,(t-1)$$

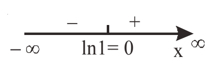
Critical point : t = 1

Sign of $\frac{dy}{dx}$ based on t :



Since, x = lnt increases with t, the order of sign of the derivative is maintained on the x number line.

Sign of $\frac{dy}{dx}$ based on x :



f(x) is increasing for $x \in (0, \infty)$ and it is decreasing for $x \in (-\infty, 0)$.

Example 3. Find the intervals of monotonicity of the function y = f(x) given by $x = 1 + 2^{-t}$, $y = 2t - t^2$, $t \in R$.

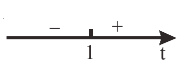
Solution $\frac{dy}{dx} = \frac{2-2t}{-2^{-t}\ln 2} = \frac{2(t-1)}{2^{-t}\ln 2}$

Critical point : t = 1

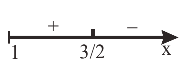
Since, $x = 1 - 2^{-t}$ decreases with t, the order of sign of the derivative is reversed on the x number line.

Note that x > 1.

Sign of $\frac{dy}{dx}$



$\Rightarrow$



f(x) is increasing for $x \in (1, 3/2)$ and f(x) is decreasing for $x \in (3/2, \infty)$.

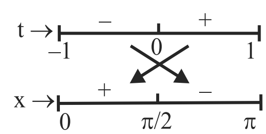
Example 4. Find the intervals of monotonicity of the function y = f(x) given by

$$x = \cos^{-1} t,\ \ y = \ln(4 - t^2),\ -1 \le t \le 1.$$

Solution $\frac{dy}{dx} = \frac{2t\sqrt{1-t^2}}{(4-t^2)}$

Critical point : t = 0

Sign of $\frac{dy}{dx}$



f(x) is increasing for $x \in (0, \pi/2)$ and f(x) is decreasing for $x \in (\pi/2, \pi)$.

6.6 ALGEBRA OF MONOTONOUS FUNCTIONS

1. Negative

If f(x) is a strictly increasing function then its negative g(x) = – f(x) is a strictly decreasing function and vice–versa.

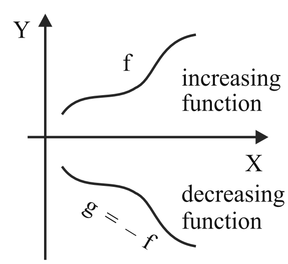
Assuming f to be differentiable, $g'(x) = -f'(x)$

Since $f'(x) > 0$, $g'(x) < 0$

$\Rightarrow$ g is a strictly decreasing function

$f(x) = \tan^{-1}x$ is strictly increasing.

$\therefore$ $g(x) = -\tan^{-1}x$ is strictly decreasing.



In short, –(an increasing function)

= a decreasing function

i.e. – I = D

Similarly, – D = I

2. Reciprocal

The reciprocal of a nonzero strictly increasing function is a strictly decreasing function and vice-versa.

In short, $\dfrac{1}{\text{an increasing function}}$ = a decreasing function

i.e. (i) $\dfrac{1}{I} = D$ (ii) $\dfrac{1}{D} = I$

Example 1. Find the intervals of monotonicity of

$$g(x) = \frac{1}{4x^3 - 9x^2 + 6x}$$

Solution Let $f(x) = 4x^3 - 9x^2 + 6x$

$g(x) = 1/f(x)$

Hence, when f increases then g decreases and vice-versa. So we first find the monotonicity of f(x).

$f'(x) = 12x^2 - 18x + 6$

$\Rightarrow 6(2x^2 - 3x + 1) = 6(2x - 1)(x - 1)$

$$\begin{array}{ccccc} + & & - & & + \\ & 1/2 & & 1 & \end{array}$$

f is strictly increasing in $(-\infty, 1/2)$, $(1, \infty)$ and strictly decreasing in $(1/2, 1)$.

and $y = 1/x$ is strictly decreasing in $(-\infty, 0)$, $(0, \infty)$

$4x^3 - 9x^2 + 6x = 0 \Rightarrow x(4x^2 - 9x + 6) = 0 \Rightarrow x = 0.$

For $x < 0$, $f(x) = x(4x^2 - 9x + 6) < 0$

$\therefore$ $g(x) = 1/f(x)$ is strictly decreasing in $(-\infty, 0)$.

For $x > 0$, $f(x) = x(4x^2 - 9x + 6) > 0$

$\therefore$ g(x) is strictly decreasing in $(0, 1/2)$ and $(1, \infty)$ and it is strictly increasing in $(1/2, 1)$.

3. Sum

If f and g are strictly increasing functions then h(x) = f(x) + g(x) is also a strictly increasing function.

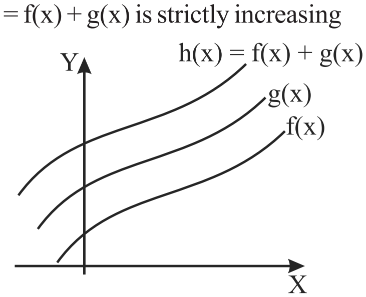
Assuming f and g to be differentiable,

$h'(x) = f'(x) + g'(x)$

Since, f and g are strictly increasing $f'(x)$ and $g'(x)$ are positive.

$\Rightarrow$ $f'(x) + g'(x)$ is positive

$\Rightarrow$ $h(x) = f(x) + g(x)$ is strictly increasing



In short, a strictly increasing function + a strictly increasing function = a strictly increasing function

i.e. (i) $I + I = I$

Similarly (ii) $D + D = D$

Note that we cannot say anything about I + D.

For example, $f(x) = \sqrt{x}$ and $g(x) = \ln x$ are strictly increasing, hence $y = \sqrt{x} + \ln x$ is also strictly increasing.

We find that $y = \sqrt{3-x} + \cos^{-1}\left(\dfrac{x-1}{2}\right)$ is strictly decreasing because $y = \sqrt{3-x}$ and $y = \cos^{-1}\left(\dfrac{x-1}{2}\right)$ are strictly decreasing.

4. Difference

Monotonicity of the difference of two function can be predicted using (1) and (3)

Consider $y = \ln(x + \sqrt{x^2+1}) - \cot^{-1}x$

increasing – decreasing

$\Rightarrow$ increasing + (–decreasing)

$\Rightarrow$ increasing + increasing $\Rightarrow$ increasing

Hence, the function is strictly increasing.

I – I = I + (–I) = I + D = cannot say anything

I – D = I + (–D) = I + I = increasing

D – I = D + (–I) = D + D = decreasing

D – D = D + (–D) = D + I = cannot say anything.

5. Product

Consider $h(x) = f(x) \times g(x)$

Case I: Both the function f and g involved in the product are positive. Further if, both f and g are strictly increasing then $h(x) = f(x) \times g(x)$ is also strictly increasing.

We have $h'(x) = f'(x)\,g(x) + f(x)\,g'(x)$

Here, all the terms in the R.H.S. are positive under the conditions given above. Hence, h(x) is strictly increasing.

In short, $I \times I = I$

$D \times D = D$

$I \times D$ = cannot say anything

Case II : If f is strictly increasing and takes negative values and g is strictly decreasing and takes positive values then

$h(x) = f(x) \times g(x)$ is strictly increasing.

$h'(x) = f'(x)\,g(x) + f(x)\,g'(x)$

Under the given conditions the R.H.S. becomes positive and hence the function h is strictly increasing.

6. Division

Monotonicity of division of two functions can be predicted by using reciprocal and product.
Assuming that both the functions I and D take positive values, we have

$$\frac{I}{D} = I \times \frac{1}{D} = I \times I = I.$$

7. Composition

If $y = g(u)$ and $u = f(x)$, then $y = (gof)(x) = g(f(x))$

and $\frac{dy}{dx} = (g(f(x)))' = g'(f(x)).f'(x)$.

(i) If f is strictly increasing in [a, b] and g is strictly increasing in [f(a), f(b)], then gof is strictly increasing in [a, b].
For example, consider $y = \tan^{-1}(e^x)$.
$\because$ $\tan^{-1}x$ is strictly increasing for all x and e^x is strictly increasing for all x, so the composite function $\tan^{-1}(e^x)$ is strictly increasing for all x.

(ii) If f is strictly decreasing in [a, b] and g is strictly decreasing in [f(b), f(a)], then gof is strictly increasing in [a, b].
For example, $y = \cot^{-1}(\log_{1/2}x)$ is strictly increasing because $\log_{1/2}x$ is strictly decreasing for all x and $\cot^{-1}x$ is also strictly decreasing for all x.

(iii) If f is strictly increasing in [a, b] and g is strictly decreasing in [f(a), f(b)], then gof is strictly decreasing in [a, b].
For example, $y = \cos(\sin^{-1}x)$ is strictly decreasing in [0, 1], because $\sin^{-1}x$ is strictly increasing in [0, 1] and cos x is strictly decreasing in $[0, \pi/2]$.

(iv) If f is strictly decreasing in [a, b] and g is strictly increasing in [f(b), f(a)], then gof is strictly decreasing in [a, b].
For example, $y = \ln(\cot^{-1}x)$ is strictly decreasing for all x because $\cot^{-1}x$ is strictly decreasing for all x and ln x is strictly increasing for all $x > 0$.

In short,

(i) I(I) = I (ii) I(D) = D
(iii) D(I) = D (iv) D(D) = I

Example 2. Find the monotonicity of

$$h(x) = e^{(\sin x + \cos x)},\ x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right].$$

Solution Let $f(x) = \sin x + \cos x$ and $g(x) = e^x$
Then $h(x) = g(f(x))$

$$f(x) = \sqrt{2}\sin(x + \pi/4)$$

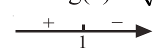
$$f'(x) = \sqrt{2}\cos(x + \pi/4) < 0 \text{ for } x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$$

Hence f is strictly decreasing while g is strictly increasing. Using the above results, h(x) is strictly decreasing in

$$\left[\frac{\pi}{4}, \frac{\pi}{2}\right].$$

Example 3. Find the intervals of monotonicity of $y = \sqrt{2x - x^2}$.

Solution Let $f(x) = 2x - x^2$ and $g(x) = \sqrt{x}$.
$\Rightarrow f'(x) = 2 - 2x$



f is strictly increasing in $(-\infty, 1)$ and strictly decreasing in $(1, \infty)$.
Domain of the given function : $2x - x^2 \geq 0$
$\Rightarrow 0 \leq x \leq 2$
g is strictly increasing for $x \geq 0$.
So, for $0 < x < 1$, f is strictly increasing and for $0 < x < 1$, g is strictly increasing. Hence gof is strictly increasing in (0, 1).
Similarly, for $1 < x < 2$, f is strictly decreasing and for $0 < x < 1$, g is strictly increasing. Hence gof is strictly decreasing in (1, 2).

Example 4. Given $f(x) = \tan^{-1}(\sin x + \cos x)^3$, find intervals of increase in $(0, 2\pi)$.

Solution Since $\tan^{-1}(\sin x + \cos x)^3$ and x^3 are both increasing functions, f(x) is an increasing function when sin x + cos x is an increasing function.
Let $g(x) = \sin x + \cos x,\ x \in (0, 2\pi)$.
We should have $g'(x) = \cos x - \sin x > 0$

$$\Rightarrow \frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x > 0 \Rightarrow \cos\left(x + \frac{\pi}{4}\right) > 0$$

then $0 < x + \frac{\pi}{4} < \frac{\pi}{2}$ and $\frac{3\pi}{2} < x + \frac{\pi}{4} < 2\pi$

$$\Rightarrow -\frac{\pi}{4} < x < \frac{\pi}{4} \text{ and } \frac{5\pi}{4} < x < \frac{7\pi}{4}$$

Hence, the intervals of increase of f(x) are

$$\left(0, \frac{\pi}{4}\right) \text{ and } \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right).$$

Example 5. Find the intervals in which the function $f(x) = \sin(\ell nx) - \cos(\ell nx)$ increases.

Solution Since ln x is an increasing function, f(x) increases when sin x – cos x increases.

Let $g(x) = \sin x - \cos x$

$$g'(x) = \cos x + \sin x = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right)$$

$$g'(x) > 0 \text{ when } 2n\pi - \frac{\pi}{2} < x - \frac{\pi}{4} < 2n\pi + \frac{\pi}{2}$$

$$\Rightarrow\ 2n\pi - \frac{\pi}{4} < 2n\pi + \frac{2\pi}{4},\ n \in I$$

Now the intervals of increase of f(x) are given by :

$$2n\pi - \frac{\pi}{4} < \ln x < 2n\pi + \frac{3\pi}{4}$$

$$e^{2n\pi - \frac{\pi}{4}} < x < e^{2n\pi + \frac{3\pi}{4}},\ n \in I.$$

Alternative : $f(x) = \sin(\ell nx) - \cos(\ell nx)$

$$= \sqrt{2}\sin\left(\ell nx - \frac{\pi}{4}\right),\ x > 0.$$

$$\therefore\ f'(x) = \frac{\sqrt{2}}{x}\cos\left(\ell nx - \frac{\pi}{4}\right)$$

$$= \frac{\sqrt{2}}{x}\sin\left(\frac{\pi}{2} + \ell nx - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{x}\sin\left(\frac{\pi}{4} + \ell nx\right) > 0$$

$$\Rightarrow\ \sin\left(\frac{\pi}{4} + \ell nx\right) > 0 \qquad (\text{since } x > 0)$$

$$\Rightarrow\ 2n\pi < \frac{\pi}{4} + \ell nx < 2n\pi + \pi,\ n \in I$$

$$\Rightarrow\ 2n\pi - \frac{\pi}{4} < \ell nx < 2n\pi + \frac{3\pi}{4},\ n \in I$$

$$\Rightarrow\ e^{2n\pi - \frac{\pi}{4}} < x < e^{2n\pi + \frac{3\pi}{4}},\ n \in I.$$

Example 6. Find the interval of monotonicity of $y = \ln\left(\frac{\ln x}{x}\right)$.

Solution Let $f(x) = \frac{\ln x}{x}$ and $g(x) = \ln x$

$$f'(x) = \frac{1 - \ln x}{x^2}$$

0 + e −

f is strictly increasing in (0, e) and strictly decreasing in (e, ∞). g is increasing for x > 0.

Domain of the given function : $\frac{\ln x}{x} > 0 \Rightarrow x > 1$.

So, for 1 < x < e, f is strictly increasing and for 0 < x < 1/e, g is strictly increasing. Hence gof is strictly increasing in (1, e).

Similarly, for e < x < ∞, f is strictly decreasing and for 1 < x < ∞, g is strictly increasing. Hence gof is strictly decreasing in (e, ∞).

6.7 PROVING INEQUALITIES

Greatest and least values of a function

If a continuous function y = f(x) is strictly increasing in the closed interval [a, b] then f(a) is the least value and f(b) is the greatest value of f(x) in [a, b]. (See figure - 1)

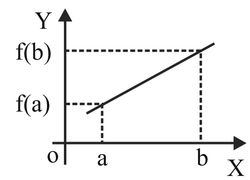


Figure - 1

If f(x) is strictly decreasing in [a, b] then f(b) is the least and f(a) is the greatest value of f(x) in [a, b].(See figure - 2)

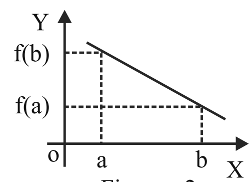


Figure - 2

However if f(x) is non-monotonic in [a, b] and is continuous then the greatest and least values of f(x) in [a, b] are found where f′(x) = 0 or f′(x) does not exist or at the endpoints. (See figure - 3)

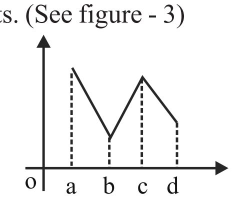


Figure - 3

Example 1. Show that

$$0 < x\sin x - \frac{1}{2}\sin^2 x < \frac{(\pi - 1)}{2}\quad \forall\ x \in \left(0, \frac{\pi}{2}\right).$$

Solution Let $f(x) = x\sin x - \frac{1}{2}\sin^2 x$

$$\Rightarrow\ f'(x) = x\cos x + \sin x - \sin x\cos x$$

$$= \sin x(1 - \cos x) + x\cos x$$

For $x \in (0, \pi/2)$, $\sin x > 0$, $1 - \cos x > 0$, $\cos x > 0$

$\Rightarrow\ f'(x) > 0\ \forall\ x \in (0, \pi/2)$.

f is strictly increasing in $(0, \pi/2)$.

$\Rightarrow$ The range of f(x) is $(\lim\limits_{x\to 0} f(x), \lim\limits_{x\to \pi/2} f(x))$

$\equiv\left(0, \dfrac{\pi-1}{2}\right)$

$\Rightarrow$ $0 < x\sin x - \dfrac{1}{2}\sin^2 x < \dfrac{\pi-1}{2}$.

Example 2. If 1/6 < x < 5/6, then prove that $\dfrac{1}{2} < 3\left(x + \dfrac{1}{2\pi} - \dfrac{\sin \pi x}{\pi}\right) < \dfrac{5}{2}$.

Solution Consider $f(x) = 3\left(x + \dfrac{1}{2\pi} - \dfrac{\sin \pi x}{\pi}\right)$.

Now, $f'(x) = 3(1 - \cos \pi x) > 0\ \forall\ x \in (1/6, 5/6)$

Applying the increasing function f on the inequality $\dfrac{1}{6} < x < \dfrac{5}{6}$, we get $f\left(\dfrac{1}{6}\right) < f(x) < f\left(\dfrac{5}{6}\right)$

i.e. $\dfrac{1}{2} < f(x) < 5/2$

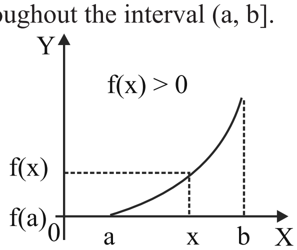
Hence, the statement is proved.

General approach in proving inequalities

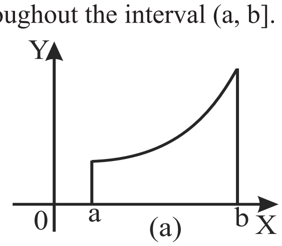
The methods of the investigation of the behaviour of functions can be applied to proving inequalities.

If $f'(a) > 0$ then $f(x) < f(a)$ for all values of x less than a but sufficiently near to a, and $f(x) > f(a)$ for all values of x greater than a but sufficiently near to a.

If f(x) is continuous in [a, b] and differentiable in (a, b) where $f'(x) > 0$ for all $x \in (a, b)$ and $f(a) \geq 0$ then f(x) is positive throughout the interval (a, b].

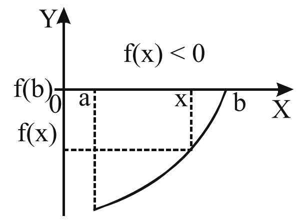


If f(x) is continuous in [a, b] and differentiable in (a, b) where $f'(x) < 0$ for all $x \in (a, b)$ and $f(a) \leq 0$ then f(x) is negative throughout the interval (a, b].

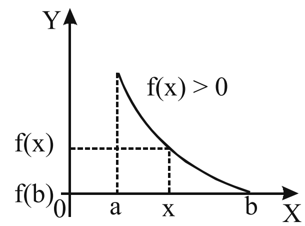


(a)

If f(x) is continuous in [a, b] and differentiable in (a, b) where $f'(x) > 0$ for all $x \in (a, b)$ and $f(b) \leq 0$ then f(x) is negative throughout the interval [a, b).



If f(x) is continuous in [a, b] and differentiable in (a, b) where $f'(x) < 0$ for all $x \in (a, b)$ and $f(b) \geq 0$ then f(x) is positive throughout the interval [a, b).



Note: If f(x) is discontinuous at the endpoints a or b then one sided limits $f(a^+)$ or $f(b^-)$ are used in place of f(a) or f(b) for understanding the sign of the function.

Thus, to prove $f(x) \geq g(x)\ \forall\ x \geq a$,

we assume $h(x) = f(x) - g(x)$, and

find $h'(x) = f'(x) - g'(x)$

If $h'(x) \geq 0\ \forall\ x \geq a$,

h is an increasing function for $x \geq a$.

Thus, $h(x) \geq h(a)$.

If $h(a) \geq 0$, then $h(x) \geq 0\ \forall\ x \geq a$

i.e. the given inequality is established.

Note: If the sign of $h'(x)$ is not obvious, then to determine its sign we assume $p(x) = h'(x)$ and apply the above procedure on p(x).

Example 3. Prove the inequality

$$\ln(1+x) > x - \frac{x^2}{2}\quad \forall\ x \in (0, \infty).$$

Solution Consider the function

$f(x) = \ln(1+x) - x + \dfrac{x^2}{2}$, $x \in (0, \infty)$

Then $f'(x) = \dfrac{1}{1+x} - 1 + x = \dfrac{x^2}{1+x} > 0\quad \forall\ x \in (0, \infty)$

$\Rightarrow$ f(x) is strictly increasing in $(0, \infty)$.

$\Rightarrow$ $f(x) > f(0)\ \forall\ x \in (0, \infty)$

$\Rightarrow$ $f(x) > 0$

i.e. $\ln(1+x) - x + \dfrac{x^2}{2} > 0, \quad x \in (0, \infty)$

$\Rightarrow$ $\ln(1+x) > x - \dfrac{x^2}{2}\ \ \forall\ x \in (0, \infty)$

which is the desired result.

Example 4. For $x \in (0, \pi/2)$, prove that $\sin x < x < \tan x$.

Solution Let $f(x) = x - \sin x$

$\Rightarrow$ $f'(x) = 1 - \cos x$

$f'(x) > 0$ for $x \in (0, \pi/2)$

$\Rightarrow$ $f(x)$ is strictly increasing for $x \in (0, \pi/2)$

$\Rightarrow$ $f(x) > f(0) \quad \Rightarrow x - \sin x > 0$

$\Rightarrow$ $x > \sin x$

Similarly consider another function, $g(x) = x - \tan x$

$\Rightarrow$ $g'(x) = 1 - \sec^2 x$

$g'(x) < 0$ for $x \in (0, \pi/2)$

$\Rightarrow$ $g(x)$ is strictly decreasing for $x \in (0, \pi/2)$

Hence, $g(x) < g(0) \Rightarrow x - \tan x < 0$

$\Rightarrow$ $x < \tan x$

Thus, $\sin x < x < \tan x$ for $x \in (0, \pi/2)$.

Example 5.

(i) Find the order relation between x and $\tan^{-1}x$.

(ii) Show that $\ln(1+x) < x$ for all $x > 0$

Solution

(i) Let $f(x) = x - \tan^{-1}x$.

$\Rightarrow$ $f'(x) = 1 - \dfrac{1}{1+x^2} = \dfrac{x^2}{1+x^2} \geq 0\ \forall x \in R$

Thus f(x) is a strictly increasing function.

Now, $f(0) = 0 \Rightarrow f(x) < 0,\ \forall\ x \in (-\infty, 0)$ and

$f(x) \geq 0,\ x \in [0, \infty)$

$\Rightarrow$ $x < \tan^{-1}x,\ x \in (-\infty, 0)$

and $x \geq \tan^{-1}x,\ \ x \in [0, \infty)$.

(ii) Let us assume $f(x) = \ln(1+x) - x$.

$\Rightarrow$ $f'(x) = \dfrac{1}{1+x} - 1 = \dfrac{-x}{1+x}$

Clearly, $f'(x) < 0\ \forall\ x \in (0, \infty)$.

Hence f(x) is strictly decreasing for $x > 0$.

Moreover $f(0) = 0$, hence $f(x) < 0$ for $x > 0$.

$\Rightarrow$ $\ln(1+x) - x < 0$

$\Rightarrow$ $\ln(1+x) < x$ for all $x > 0$.

Example 6. Prove the inequality $2\sin x + \tan x \geq 3x$ for $x \in [0, \pi/2)$.

Solution Consider $f(x) = 2\sin x + \tan x - 3x$

$f'(x) = 2\cos x + \sec^2 x - 3$

$= \dfrac{2\cos^3 x - 3\cos^2 x + 1}{\cos^2 x}$

$= \dfrac{(\cos x - 1)^2 (2\cos x + 1)}{\cos^2 x}$

Hence $f'(0) > 0$ for $x \in (0, \pi/2)$.

$\Rightarrow$ $f(x)$ is strictly increasing for $x \in (0, \pi/2)$.

$\therefore$ $f(x) > f(0)$ for $x \in (0, \pi/2)$

Since $f(0) = 0$, $f(x) \geq 0$ for $x \in [0, \pi/2)$.

Hence $f(x) \geq 0$ for $x \in [0, \pi/2)$.

$\Rightarrow$ $2\sin x + \tan x \geq 3x$ for $x \in [0, \pi/2)$.

Example 7. For $x \in (0, \pi/2)$, prove that

$\sin x > x - \dfrac{x^3}{6}$.

Solution Let $f(x) = \sin x - x + \dfrac{x^3}{6}$.

$\Rightarrow$ $f'(x) = \cos x - 1 + \dfrac{x^2}{2}$

It is difficult to decide at this point whether $f'(x)$ is positive or negative, hence let us check for monotonic behaviour of $f'(x)$

$f''(x) = x - \sin x$

Since $f''(x) > 0$ for $x \in (0, \pi/2)$, we conclude that

$f'(x)$ is strictly increasing for $x \in (0, \pi/2)$.

$\Rightarrow$ $f'(x) > f(0) \Rightarrow f'(x) > 0$

$\Rightarrow$ $f(x)$ is strictly increasing for $x \in (0, \pi/2)$

$\Rightarrow$ $f(x) > f(0)$ for $x \in (0, \pi/2)$

$\Rightarrow$ $\sin x - x + \dfrac{x^3}{6} > 0$

$\Rightarrow$ $\sin x > x - \dfrac{x^3}{6}$.

Example 8. Establish the inequality

$\dfrac{2}{2x+1} < \ell n\left(1 + \dfrac{1}{x}\right) < \dfrac{1}{x}$ for $x > 0$.

Solution Consider $f(x) = \dfrac{2}{2x+1} - \ell n\left(1 + \dfrac{1}{x}\right)$

$f'(x) = \dfrac{-4}{(2x+1)^2} - \dfrac{1}{1 + (1/x)} \cdot \left(-\dfrac{1}{x^2}\right)$

$= \dfrac{1}{x(x+1)} - \dfrac{4}{(2x+1)^2} = \dfrac{1}{x(x+1)(2x+1)^2}$

which is always positive for $x > 0$

Hence f(x) is strictly increasing for $x > 0$.

$\therefore\ f(x) < \lim_{x\to\infty} f(x)$

$\Rightarrow\ f(x) < \lim_{x\to\infty}\left\{\frac{2}{2x+1} - \ln\left(1+\frac{1}{x}\right)\right\} = 0.$

So, $f(x) < 0$

$\Rightarrow\ \frac{2}{2x+1} < \ell n\left(1+\frac{1}{x}\right)$

Similarly, consider $g(x) = \ell n\left(1+\frac{1}{x}\right) - \frac{1}{x}$.

$$g'(x) = \frac{1}{x^2} - \frac{1}{x(x+1)} = \frac{1}{x^2(x+1)}$$

$g'(x)$ is strictly increasing for $x > 0$.

So, $g(x) < \lim_{x\to\infty} g(x)$

$\Rightarrow\ g(x) < 0.$

Example 9. Prove that, $2x\sec x + x > 3\tan x$ for $0 < x < \pi/2$.

Solution $f(x) = 2x\sec x + x - 3\tan x$

$f'(x) = 2\sec x + 2x\sec x\ \tan x + 1 - 3\sec^2 x$

$= \sec^2 x\ [2\cos x + 2x\sin x + \cos^2 x - 3]$

Consider $g(x) = 2\cos x + 2x\sin x + \cos^2 x - 3$

$g'(x) = -2\sin x + 2x\cos x + 2\sin x - 2\sin x\cos x$

$= 2\cos x\,(x - \sin x) > 0$ for $x \in (0, \pi/2)$.

Hence, $g(x) > g(0) \Rightarrow g(x) > 0$.

Now, $f'(x) > 0$

Hence, $f(x) > f(0) \Rightarrow f(x) > 0$

Thus, $2x\sec x + x > 3\tan x$ for $0 < x < \pi/2$.

Example 10. Show that

$$1 + x\ln(x + \sqrt{x^2+1}) \ge \sqrt{1+x^2} \text{ for all } x \ge 0.$$

Solution Let $f(x) = 1 + x\ln(x + \sqrt{x^2+1}) - \sqrt{1+x^2}$

$$\therefore\ f'(x) = \frac{x}{[x+\sqrt{x^2+1}]}\cdot\left[1 + \frac{1.(2x)}{2\sqrt{x^2+1}}\right] + \ln(x+\sqrt{x^2+1}).1 - \frac{1(2x)}{2\sqrt{1+x^2}}$$

$$\Rightarrow\ f'(x) = \frac{x}{(x+\sqrt{x^2+1})}\left[\frac{\sqrt{x^2+1}+x}{2\sqrt{x^2+1}}\right] + \ln(x+\sqrt{x^2+1}) - \frac{x}{\sqrt{1+x^2}}$$

$$\Rightarrow\ f'(x) = \frac{x}{\sqrt{x^2+1}} + \ln(x+\sqrt{x^2+1}) - \frac{x}{\sqrt{1+x^2}}$$

$\Rightarrow\ f'(x) = = \ln(x + \sqrt{x^2+1})$

We have $x \ge 0$, so, $\ln(x+\sqrt{x^2+1}) \ge 0$

$\Rightarrow\ f'(x) \ge 0 \quad \Rightarrow$ f(x) is increasing for $x \ge 0$

$\Rightarrow\ f(x) \ge f(0)$

$\Rightarrow\ 1 + x\ln(x+\sqrt{x^2+1}) - \sqrt{1+x^2} \ge 1 + 0 - \sqrt{1}$

$\Rightarrow\ 1 + x\ln(x + \sqrt{x^2+1}) \ge \sqrt{1+x^2}$ for $x \ge 0$.

Example 11. Examine which is greater $\sin x\tan x$ or x^2. Hence evaluate $\lim_{x\to 0}\left[\frac{\sin x\tan x}{x^2}\right]$, where $x \in \left(0, \frac{\pi}{2}\right)$

Solution Let $f(x) = \sin x \cdot \tan x - x^2$

$f'(x) = \cos x \cdot \tan x + \sin x \cdot \sec^2 x - 2x$

$\Rightarrow\ f'(x) = \sin x + \sin x\sec^2 x - 2x$

$\Rightarrow\ f''(x) = \cos x + \cos x\sec^2 x + 2\sec^2 x\sin x\tan x - 2$

$\Rightarrow\ f''(x) = (\cos x + \sec x - 2) + 2\sec^2 x\sin x\tan x$

Now $\quad \cos x + \sec x - 2 = (\sqrt{\cos x} - \sqrt{\sec x})^2$

and $2\sec^2 x\tan x \cdot \sin x > 0$ because $x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow\ f''(x) > 0 \quad \Rightarrow f'(x)$ is strictly increasing.

Hence $\quad f'(x) > f'(0)$

$\Rightarrow\ f'(x) > 0 \quad \Rightarrow$ f(x) is strictly increasing

$\Rightarrow\ f(x) > 0 \quad \Rightarrow\ \sin x\tan x - x^2 > 0$

Hence, $\sin x\tan x - x^2$

$\Rightarrow\ \frac{\sin x\tan x}{x^2} > 1$

$\Rightarrow\ \lim_{x\to 0}\left[\frac{\sin x\tan x}{x^2}\right] = 1.$

Example 12. Prove the inequalities.

$$\frac{2}{\pi}x < \sin x < x \text{ for } 0 < x < \frac{\pi}{2}.$$

Solution We introduce the function

$$f(x) = \frac{\sin x}{x},\quad x \ne 0$$

Its derivative $f'(x) = \frac{\cos x}{x^2}(x - \tan x)$

is negative in the interval $\left(0, \frac{\pi}{2}\right)$ since $x < \tan x$.

Thus, f(x) is a decreasing function in that interval.

$$f(0^+) = 1 \text{ and } f\left(\frac{\pi}{2}\right) = \frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} = \frac{2}{\pi}$$

Hence, for $0 < x < \frac{\pi}{2}$, the values of the expression $\frac{\sin x}{x}$ are less than the value 1 of the function reached at the point x = 0 and exceed the value of the function attained at the point $x = \frac{\pi}{2}$.

Thus, $f\left(\frac{\pi}{2}\right) < f(x) < f(0^+)$

$\Rightarrow \quad \frac{2}{\pi} < \frac{\sin x}{x} < 1.$

Example 13. If P(1) = 0 and $\frac{d}{dx}\{P(x)\} > P(x)$ for all $x \geq 1$, then prove that P(x) > 0 for all x > 1.

Solution Here, $\frac{d}{dx}\{P(x)\} > P(x)$ for all $x \geq 1$

$\Rightarrow \quad e^{-x}\frac{dP(x)}{dx} > P(x)e^{-x}$, for all $x \geq 1$ since $e^{-x} > 0$

$\Rightarrow \quad e^{-x}\frac{dP(x)}{dx} - P(x)e^{-x} > 0$, for all $x \geq 1$

$\Rightarrow \quad \frac{d}{dx}\{P(x)e^{-x}\} > 0$, for all $x \geq 1$

$\Rightarrow \quad P(x)e^{-x}$ is an increasing function for all $x \geq 1$

$\Rightarrow \quad P(x)e^{-x} > P(1)e^{-1}$ for all x > 1

$\Rightarrow \quad P(x)e^{-x} > 0$ for all x > 1 {since P(1) = 0}

Thus, P(x) > 0 for all x > 1 {since $e^{-x} > 0$ for all x}.

Inequalities based on composite functions

Example 14. For $x \in (0, \pi/2)$, prove that
cos(sin x) > sin(cos x).

Solution Let f(x) = x – sin x

$\therefore \quad f'(x) = 1 - \cos x > 0,\ x \in (0, \pi/2)$.

Hence f(x) in increasing in $(0, \pi/2)$

Then f(x) > f(0)

or $\quad x - \sin x > 0 \Rightarrow \quad x > \sin x$ ...(1)

Again $x \in (0, \pi/2)$ and $0 < \cos x < 1$, therefore using (1)

$\quad \cos x > \sin(\cos x)$ ...(2)

Now using (1) and the fact that cosx is decreasing in $(0, \pi/2)$

$\therefore \quad \cos x < \cos(\sin x)$ ...(3)

From (2) and (3) we get

$\quad \sin(\cos x) < \cos x < \cos(\sin x)$

Hence, sin(cos x) < cos(sin x).

Example 15. Prove that
$\sin^2\theta < \theta < \sin(\sin\theta)$ for $\theta \in (0, \pi/2)$

Solution We have to prove

$\sin^2\theta < \theta\sin(\sin\theta)$ or $\frac{\sin\theta}{\theta} < \frac{\sin(\sin\theta)}{\sin\theta}$

Now let $f(x) = \frac{\sin x}{x}$, $x \in (0, \pi/2)$

$\therefore \quad f'(x) = \frac{(x\cos x - \sin x)}{x^2} = \frac{\cos x(x - \tan x)}{x^2}$

which is negative since $x - \tan x < 0$ in $x \in (0, \pi/2)$.

$\Rightarrow \quad$ f(x) is a decreasing function

$\because \quad \sin\theta < \theta$ for $x \in (0, \pi/2)$ we have
$\quad f(\sin\theta) > f(\theta)$

$\Rightarrow \quad \frac{\sin(\sin\theta)}{\sin\theta} > \frac{\sin\theta}{\theta}$

Hence $\sin^2\theta < \theta\sin(\sin\theta)$ for $x \in (0, \pi/2)$.

Example 16. Prove that sin 1 > cos (sin 1). Also show that the equation sin (cos (sin x)) = cos (sin (cos x)) has only one solution in $x \in (0, \pi/2)$.

Solution $\sin 1 > \cos(\sin 1)$ if $\cos\left(\frac{\pi}{2} - 1\right) > \cos(\sin 1)$

i.e. if $\frac{\pi}{2} - 1 < \sin 1$

i.e. if $\sin 1 > \left(\frac{\pi - 2}{2}\right)$ ...(1)

We have $\sin 1 > \sin\frac{\pi}{4} > \frac{1}{\sqrt{2}} > \left(\frac{\pi - 2}{2}\right)$.

Hence (1) is true $\Rightarrow$ sin 1 > cos (sin 1).

Now let f(x) = sin (cos (sin x)) – cos (sin (cos x))

$f'(x) = -\cos(\cos(\sin x))\sin(\sin x)\cos x$
$\quad - \sin(\sin(\cos x))\cos(\cos x) \,.\, \sin x$

$\Rightarrow \quad f'(x) < 0\ \forall\ x \in (0, \pi/2)$

$\Rightarrow$ f(x) is decreasing in $(0, \pi/2)$

$\quad$ and f(0) = sin 1 – cos (sin 1) > 0

$\quad f\left(\frac{\pi}{2}\right) = \sin(\cos(1)) - 1 > 0$

Since f(0) is positive and $f\left(\frac{\pi}{2}\right)$ is negative.

f(x) = 0 has one solution in $x \in (0, \pi/2)$.

Example 17. Show that $\ln x < x\ \forall\ x > 0$. Hence, prove that $\ln(\cos\theta) < \cos(\ln\theta)$ where $e^{-\pi/2} < \theta < \frac{\pi}{2}$.

Solution Consider the function

$\quad f(x) = \ln x - x,\ x > 0$

Then $f'(x) = \dfrac{1}{x} - 1 = \dfrac{1-x}{x}$

Sign of $f'(x)$ $\quad \overset{+}{\underset{0}{|}}\!\!\longrightarrow\!\!\overset{-}{\underset{1}{|}}\!\!\longrightarrow$

$\Rightarrow$ $f(x)$ is strictly increasing in $(0, 1)$, and strictly decreasing in $(1, \infty)$.

$\Rightarrow$ $f(x)$ has greatest value at $x = 1$.

$\Rightarrow$ $f(x) \le f(1) = -1 < 0$

i.e. $\ln x < x$.

Now, we have

$$e^{-\pi/2} < \theta < \frac{\pi}{2}$$

i.e. $0 < \theta < \dfrac{\pi}{2}$

i.e. $0 < \cos\theta < 1$

i.e. $\ln(\cos\theta) < 0$ [$\because \ln x < 0\ \forall\ x \in (0, 1)$] ...(1)

Also, we have

$$e^{-\pi/2} < \theta < \frac{\pi}{2}$$

i.e. $\dfrac{-\pi}{2} < \ln\theta < \ln\dfrac{\pi}{2}$ [$\because \ln x$ is increasing in $(0, \infty)$]

i.e. $\dfrac{-\pi}{2} < \ln\theta < \dfrac{\pi}{2}$ [using $\ln x < x$] ...(2)

i.e. $0 < \cos(\ln\theta) < 1$

From results (1) and (2), we can infer that $\ln(\cos\theta) < \cos(\ln\theta)$.

Example 18. Using the relation $2(1 - \cos x) < x^2, x \ne 0$ or otherwise prove that $\sin(\tan x) \ge x$ for all $x \in \left[0, \dfrac{\pi}{4}\right]$

Solution Let $f(x) = \sin(\tan x) - x$

Then, $f'(x) = \cos(\tan x) \cdot \sec^2 x - 1$

$\Rightarrow$ $f'(x) = \cos(\tan x)(1 + \tan^2 x) - 1$

Using the relation $2(1 - \cos x) < x^2$ i.e. $\cos x > 1 - \dfrac{x^2}{2}$

We have $\cos(\tan x) > 1 - \dfrac{\tan^2 x}{2}$

Hence, $f'(x) > \left(1 - \dfrac{\tan^2 x}{2}\right)(1 + \tan^2 x) - 1$

$f'(x) > 1/2 \tan^2 x(1 - \tan^2 x)$

$\Rightarrow$ $f'(x) \ge 0,\ \forall\ x \in \left[0, \dfrac{\pi}{4}\right]$

$\Rightarrow$ $f(x)$ is an increasing function for all $x \in \left[0, \dfrac{\pi}{4}\right]$

$\therefore$ $f(x) \ge f(0)$, for all $x \in \left[0, \dfrac{\pi}{4}\right]$

$\Rightarrow$ $\sin(\tan x) - x > \sin(\tan 0) - 0$

$\Rightarrow$ $\sin(\tan x) \ge x$ for all $x \in \left[0, \dfrac{\pi}{4}\right]$.

Comparision of constants

Example 19. Compare which of the two is greater $(100)^{1/100}$ or $(101)^{1/101}$.

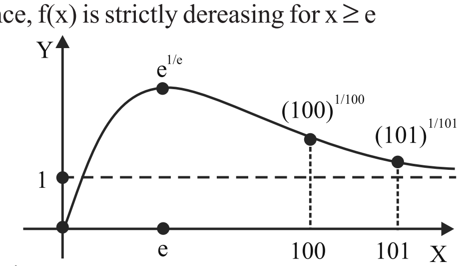
Solution Assume $f(x) = x^{1/x}$ and let us examine monotonic behaviour of $f(x)$

$$f'(x) = x^{1/x} \cdot \left(\frac{1 - \ell nx}{x^2}\right)$$

$f'(x) > 0 \Rightarrow x \in (0, e)$

and $f'(x) < 0 \Rightarrow x \in (e, \infty)$

Hence, $f(x)$ is strictly dereasing for $x \ge e$



and since $e < 100 < 101$

$\Rightarrow$ $f(100) > f(101)$

$\Rightarrow$ $(100)^{1/100} > (101)^{1/101}$.

Example 20. Prove that

$$\left(\tan^{-1}\frac{1}{e}\right)^2 + \frac{2e}{\sqrt{(e^2+1)}} < (\tan^{-1} e)^2 + \frac{2}{\sqrt{(e^2+1)}}$$

Solution We have to prove

$$\left(\tan^{-1}\frac{1}{e}\right)^2 + \frac{2e}{\sqrt{(e^2+1)}} < (\tan^{-1} e)^2 + \frac{2}{\sqrt{(e^2+1)}}$$

or
$$\left(\tan^{-1}\frac{1}{e}\right)^2 + \frac{2}{\sqrt{\left(\frac{1}{e}\right)^2 + 1}} < (\tan^{-1} e)^2 + \frac{2}{\sqrt{e^2+1}}$$

Now let $f(x) = (\tan^{-1} x)^2 + \dfrac{2}{\sqrt{x^2+1}}$ ...(1)

$$\therefore \quad f'(x) = \frac{2\tan^{-1} x}{(1+x^2)} - \frac{2x}{(x^2+1)^{3/2}}$$

$$= \frac{2}{(1+x^2)}\left\{\tan^{-1} x - \frac{x}{\sqrt{(x^2+1)}}\right\} \quad ...(2)$$

To find sign of $f'(x)$ we consider

$$g(x) = \tan^{-1}x - \frac{x}{\sqrt{(x^2+1)}}$$

$$\therefore \quad g'(x) = \frac{1}{(1+x^2)}\left\{1 - \frac{1}{\sqrt{(x^2+1)}}\right\} > 0$$

$\Rightarrow \quad g'(x) > 0$

$\therefore \quad g(x)$ is an increasing function

Thus, $g(x) > g(0) \qquad \Rightarrow g(x) > 0 \qquad ...(3)$

Using (2) and (3), $f'(x) > 0$

$\Rightarrow \quad f(x)$ is increasing function

Since $\frac{1}{e} < e$, we have $f\left(\frac{1}{e}\right) < f(e)$

$$\Rightarrow \quad \left(\tan^{-1}\frac{1}{e}\right)^2 + \frac{2}{\sqrt{\left(\frac{1}{e^2}+1\right)}} < (\tan^{-1}e)^2 + \frac{2}{\sqrt{(e^2+1)}}$$

$$\text{Hence,} \quad \left(\tan^{-1}\frac{1}{e}\right)^2 + \frac{2e}{\sqrt{e^2+1}} < (\tan^{-1}e)^2 + \frac{2}{\sqrt{(e^2+1)}}.$$

Example 21. Using the function

$$f(x) = 2x - \tan^{-1}x - \ln\left(x + \sqrt{1+x^2}\right),$$

prove that $\ln\left(\frac{2+\sqrt{3}}{\sqrt{3}}\right) < \frac{4}{\sqrt{3}} - \frac{\pi}{6}$.

Solution $f'(x) = 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}} \geq 0 \; \forall x \in R$

and equality holds at $x = 0$ only.

So, $f(x)$ is increasing in $(-\infty, \infty)$.

Since $\frac{\pi}{6} < \frac{\pi}{3}$ and f is an increasing function we have

$$f\left(\frac{\pi}{6}\right) < f\left(\frac{\pi}{3}\right).$$

$$\text{Hence,} \quad \frac{2}{\sqrt{3}} - \frac{\pi}{6} - \ln\sqrt{3} < 2\sqrt{3} - \frac{\pi}{3} - \ln\left(\sqrt{3}+2\right).$$

$$\text{Thus,} \quad \ln\left(\frac{2+\sqrt{3}}{\sqrt{3}}\right) < \frac{4}{\sqrt{3}} - \frac{\pi}{6}.$$

Example 22. Prove that for $0 \leq p \leq 1$ and for any positive a and b the inequality $(a+b)^p \leq a^p + b^p$ is valid.

Solution By dividing both sides of the inequality by b^p we get

$$\left(\frac{a}{b}+1\right)^p \leq \left(\frac{a}{b}\right)^p + 1$$

or $\quad (1+x)^p \leq 1 + x^p, \qquad ...(1)$

where $x = \frac{a}{b}$.

Let us show that the inequality (1) holds true at any positive x. Consider the function

$$f(x) = 1 + x^p - (1+x)^p, \; x \geq 0.$$

The derivative of this function

$$f'(x) = px^{p-1} - p(1+x)^{p-1} = p\left[\frac{1}{x^{1-p}} - \frac{1}{(1+x)^{1-p}}\right]$$

is positive everywhere, since, by hypothesis, $1 - p \geq 0$ and $x > 0$.

Hence, the function increases in the interval $[0, \infty)$, i.e. $f(x) = 1 + x^p - (1+x)^p > f(0) = 0$,

Thus, $1 + x^p > (1+x)^p$, which completes the proof.

If we put $p = 1/n$, then we obtain

$$\sqrt[n]{a+b} \leq \sqrt[n]{a} + \sqrt[n]{b}, \; (n \geq 1).$$

Inequalities based on non-monotonous functions

Example 23. Prove that the function

$$f(x) = -2x^3 + 21x^2 - 60x + 41$$

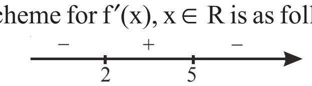
is strictly positive in the interval $(-\infty, 1)$.

Solution $f(x) = -2x^3 + 21x^2 - 60x + 41$

$\therefore \quad f'(x) = -6x^2 + 42x - 60$

$\qquad = -6(x^2 - 7x + 10) = -6(x-5)(x-2).$

The sign scheme for $f'(x)$, $x \in R$ is as follows:



$\therefore \quad$ For $x \in (-\infty, 1)$ $f(x)$ is strictly decreasing.

So when $x \in (-\infty, 1)$, $f(x) > f(1)$.

We have $f(1) = -1 + 21 - 60 + 41 = 0$.

$\therefore \quad$ For $x \in (-\infty, 1)$ we have $f(x) > 0$.

$\therefore \quad f(x)$ is strictly positive in the interval $(-\infty, 1)$.

Example 24. Prove $1 + \cot x \leq \cot\frac{x}{2} \; \forall x \in (0, \pi)$.

Solution Consider the function

$$f(x) = \cot\left(\frac{x}{2}\right) - 1 - \cot x, \; x \in (0, \pi).$$

$$\text{Then } f'(x) = \frac{-1}{2}\operatorname{cosec}^2\left(\frac{x}{2}\right) + \operatorname{cosec}^2 x$$

$$= \frac{1}{\sin^2 x} - \frac{1}{2\sin^2(x/2)}$$

$$= \frac{1}{2\sin^2(x/2)}\left[\frac{1}{2\cos^2(x/2)} - 1\right]$$

$$= \frac{-\cos x}{4\sin^2(x/2)\cos^2(x/2)}$$

$$= \frac{-\cos x}{\sin^2 x} < 0 \ \forall\, x \in (0, \pi/2)$$

$$> 0 \ \forall\, x \in (\pi/2, \pi).$$

$\Rightarrow$ f(x) strictly decreases in $(0, \pi/2)$ and strictly increases in $(\pi/2, \pi)$.

$\Rightarrow$ f(x) has least value at $x = \pi/2$.

$\Rightarrow$ $f(x) \geq f(\pi/2) = 0$

i.e. $\cot\left(\frac{x}{2}\right) \geq 1 + \cot x$

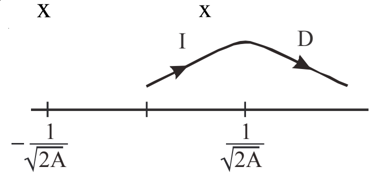
which proves the desired result.

Example 25. Find the smallest positive constant A such that $\ln x \leq Ax^2$ for all $x > 0$.

Solution Note that $A > 0$

Consider the function $f(x) = \ln x - Ax^2$

$$f'(x) = \frac{1}{x} - 2Ax = \frac{1 - 2Ax^2}{x}$$



$\Rightarrow$ $f'(x) > 0$ for $0 < x < \frac{1}{\sqrt{2A}}$

and $f'(x) < 0$ for $x > \frac{1}{\sqrt{2A}}$

$\therefore$ $\ln x < Ax^2$ for $x > 0$

This must also hold good for $x = \frac{1}{\sqrt{2A}}$

$$\therefore \ \ln \frac{1}{\sqrt{2A}} < A \cdot \frac{1}{2A}$$

$$\Rightarrow -\ln\sqrt{2A} < \frac{1}{2} \Rightarrow \ln\sqrt{2A} > -\frac{1}{2}$$

$$\sqrt{2A} = e^{-\frac{1}{2}} \Rightarrow A > \frac{1}{2e}$$

$\therefore$ The least value of $A = \frac{1}{2e}$.

Example 26. Find the least natural number a for which

$$x + ax^{-2} > 2 \ \forall\, x \in (0, \infty).$$

Solution Let $f(x) = x + ax^{-2}$

$$f'(x) = 1 - 2ax^3 = 0 \Rightarrow x = (2a)^{-1/3}$$

$f''(x) = 6ax^4 > 0 \ \forall x \in (0, \infty)$ (as a is a natural number)

Thus, $(2a)^{1/3} + a(2a)^{-2/3} > 2$

$\Rightarrow a > \frac{32}{27} \Rightarrow$ least natural number $a = 2$.

***Alternative*:**

$$x + ax^{-2} > 2 \Rightarrow x^2 - 2x^2 + a > 0$$

Let $f(x) = x^3 - 2x^2 + a$

Since $f(x) > 0 \ \forall x \in (0, \infty) \Rightarrow \min f(x) > 0$

For minimum f(x), $f'(x) = 3x^2 - 4x = 0 \Rightarrow x = 0, 4/3$

$$f(4/3) > 0 \Rightarrow a > \frac{32}{27}.$$

The least natural number a for which $x + ax^{-2} > 2$ is 2.

Example 27. If $ax^2 + \frac{b}{x} \geq c \ \forall\, x \in R^+$ where a, b, c are positive constants then prove that $27ab^2 \geq 4c^3$.

Solution Let $f(x) = ax^2 + \frac{b}{x} - c$

$$\therefore f'(x) = 2ax - \frac{b}{x^2}$$

If $f'(x) = 0$ then $x = \left(\frac{b}{2a}\right)^{1/3}$, which is a positive critical point. We can find that the least value of f(x) occurs at $x = \left(\frac{b}{2a}\right)^{1/3}$.

Since $ax^2 + \frac{b}{x} \geq c \ \forall x \in R^+$ we should have $f\left(\frac{b}{2a}\right)^{1/3} \geq 0$.

$$\Rightarrow \ a\left\{\frac{b}{2a}\right\}^{2/3} + \frac{b}{\left\{\frac{b}{2a}\right\}^{1/3}} \geq c$$

$$\Rightarrow \ a\left(\frac{b}{2a}\right) + b \geq c\left(\frac{b}{2a}\right)^{1/3}$$

$$\Rightarrow \ \left(\frac{3b}{2}\right)^3 \geq \frac{b}{2a}.c^3 \quad \Rightarrow \frac{27b^3}{8} \geq \frac{b}{2a}.c^3$$

or $27b^2a \geq 4c^3$ $\Rightarrow 27ab^2 \geq 4c^3$.

Concept Problems D

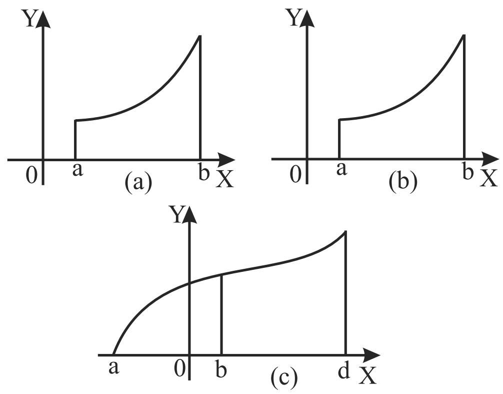
1. Prove that $f(x) = \sin(\cos x)$ in $(0, \pi/2)$ is strictly decreasing and $g(x) = \cos(\cos x)$ in $(0, \pi/2)$ is strictly increasing.
2. Let f and g be strictly increasing functions on the interval [a, b].
   (i) If $f(x) > 0$ and $g(x) > 0$ on [a, b] show that the product fg is also strictly increasing on [a, b]
   (ii) If $f(x) < 0$ and $g(x) < 0$ on [a, b], is fg strictly increasing, strictly decreasing, or neither ? Explain.
3. Suppose that f is continuous on [a, b] and that c is an interior point of the interval. Show that if $f'(x) \le 0$ on [a, c) and $f'(x) \ge 0$ on (c, b], then f(x) in never less than f(c) on [a, b].
4. Suppose that $f'(x) \ge 0$ on (a, b) and $f'(c) > 0$ for some c. Prove that $f(b) > f(a)$.
5. Suppose $f'(x) \ge 0$ on (a, b) and $f(a) = f(b)$. Prove that f is constant.
6. Suppose $f'(x) \ge g'(x)$ on (a, b) and $f(a) = g(a)$. Prove that $f(x) \ge g(x)$ on [a, b]. Further, if $f(b) = g(b)$, then prove that $f = g$.
7. Let f be differentiable at every value of x and suppose that $f(1) = 1$, that $f' < 0$ on $(-\infty, 1)$, and that $f' > 0$ on $(1, \infty)$.
   (i) Show that $f(x) \ge 1$ for all x.
   (ii) Must $f'(1) = 0$? Explain.
8. Suppose that f is differentiable on [a, b] and that $f(b) < f(a)$. Can you then say anything about the values of $f'(x)$ on [a, b]?
9. Prove that the following functions
   (i) $y = e^x - 1 - x$,
   (ii) $y = e^{-x} - 1 + x$,
   (iii) $y = 1 - x^2/2 + x^3/3 - (1 + x) e^{-x}$ are positive and increase steadily for positive x.
10. Prove that $2x > 3 \sin x - x \cos x$ for all $x \in (0, \pi/2)$.

Practice Problems D

11. (i) Suppose f is strictly increasing for $x > 0$. Let n be a positive integer and define $g(x) = x^n f(x)$ for $x > 0$. If $f(x) > 0$ on an interval [a, b] with $a > 0$, show that g is strictly increasing on the same interval.
    (ii) Let $g(x) = x^n \sin x$ for $0 < x < \frac{\pi}{2}$ where n is a positive integer. Explain why g is strictly increasing on this interval.
12. Prove the inequality
    $2\sqrt{x} > 3 - \frac{1}{x}, x > 1$
13. Prove that $\sin x + \tan x > 2x$, $0 < x < \pi/2$
14. Prove the validity of the following inequalities :
    (i) $x - \frac{x^2}{2} < \ln(1 + x) < x$ for $x > 0$,
    (ii) $\ln(1 + x) > \frac{x}{1 + x}$ for $x > 0$,
15. Prove $\sin x < x < 1 - \cos x + \sin x$ for $0 < x < \frac{1}{2}\pi$
16. Prove that the expression $(x - 1) e^x + 1$ is positive for all positive value of x.
17. Show that $2x \tan^{-1} x \ge \ln(1 + x^2)$ for all $x \in R$.
18. Show that $e^x - 1 > (1 + x).\ln(1 + x)$, if $x > 0$.
19. Show that $\tan^{-1}x > \dfrac{x}{1+\dfrac{x^2}{3}}$, if $x \in (0, \infty)$.
20. Prove that cos x lies between $1 - \frac{1}{2}x^2$ and $1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$
21. Prove that, if $x^2 < 1$, $\ln(1 + x)$ lies between $x - \frac{1}{2}x^2$ and $x - \frac{1}{2}x^2 + \frac{1}{3}x^3$.
22. Find the set of values of x for which $ln(1 + x) > \frac{x}{1 + x}$

6.8 CONCAVITY AND POINT OF INFLECTION

The figure represents the graphs of three functions each of which increases on the interval [a, b], but the difference in their behaviour is obvious.



In (a) the graph of the function is "bending downward"; in (b) it is "bending upward"; in (c) it is bending upward on the interval (a, c) and downward on the interval (c, b). From the geometrical point of view, the meaning of the expressions "bending downward" and "bending upward" is quite clear. Let us now attach a strict mathematical sense to these expressions and give a criterion for finding out in which direction the graph of a function is bending.

A curve is said to be concave up at a point P when in the immediate neighbourhood of P it lies wholly above the tangent at P. Similarly, it is said to be concave down when in the immediate neighbourhood of P it lies wholly below the tangent at P.

An arc of a curve is said to be concave up or down if it lies entirely on one side of the tangent drawn through any point of the arc.

It is of course supposed here that tangent can be drawn at each point of the arc, that is, there are neither corner points nor cusps on it.

The definition of concavity is demonstrated in the figures below. Figure 1 represents a curve which is concave up and Figure 2 represents a concave down curve. The curve in Figure 3 is neither concave up nor concave down, since it does not lie on one side of the tangent passing through the point P.

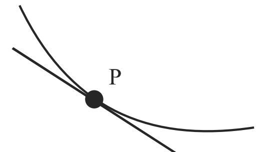


Figure 1

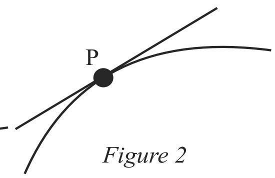


Figure 2

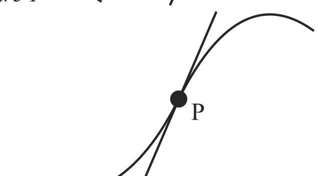


Figure 3

Point of inflection

A point of a curve separating its concave up arc from a concave down arc is termed as point of inflection.
At a point of inflection, the tangent intersects the curve.
In the vicinity of such a point the curve lies on both sides of its tangent drawn through that point.

Definition. Let f be a function and let c be a number. Assume that there are numbers a and b such that $a < c < b$ and

(i) f is continuous on the open interval (a, b),

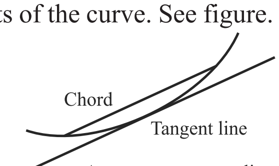
(ii) f has a tangent at the point (c, f(c)),

(iii) f is concave up in the interval (a, b) and concave down in the interval (c, b), or vice versa.

Then the point (c, f(c)) is called an inflection point or point of inflection. The number a is called an inflection number.
The point P in Figure 3 is a point of inflection.
We also speak about concavity of curves in a different way : an arc concave up (concave down) is said to be convex down (convex up).
Finally, a concave down (i.e. convex up) arc is sometimes briefly called convex while a concave up (i.e. convex down) arc is simply referred to as concave. In what follows we shall use the first variant of the terminology.

Alternative definition

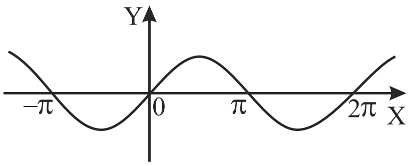
The graph of the function y = f(x) is said to be concave up on the interval (a, b) if it lies below all chords joining two points of the curve. See figure.



A concave up curve lies above its tangent lines and below its chords

Similarly, the graph of the function y = f(x) is said to be concave down on the interval (a, b) if it lies above all chords joining two points of the curve.

The curve $y = \sin x$ is concave up for $x \in (2k\pi, (2k+1)\pi)$, $k \in I$, and concave down for $x \in ((2k-1)\pi, 2k\pi)$, $k \in I$.



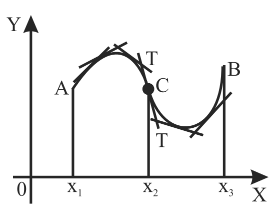
Testing a curve for concavity

It turns out that the criteria for determining the type of concavity (up or down) bring into play the second derivative – and it is the sign of the second derivative which is the discriminating device.

We consider a curve $y = f(x)$ where the function $f(x)$ is continuous together with its first and second derivatives $f'(x)$ and $f''(x)$. There is a simple relationship between the properties of the second derivative $f''(x)$ and the concavity of the curve $y = f(x)$ which we shall establish without a rigorous proof by resorting only to some simple geometrical considerations. These considerations are based on the following proposition.

On an interval of increase of the first derivative the graph of the function is concave up and on the interval of decrease of the first derivative it is concave down.

For, if the derivative $f'(x)$ increases, the slope of the tangent line to the curve also increases as the variable point of the graph traces it from left to right; then the tangent line turns as if it "lifted up" the curve preventing it from descending below the tangent. Therefore, the corresponding arc must lie above its every tangent and thus is concave. This situation is clearly demonstrated by the example of the arc CB of the curve depicted above.



In the figure arc CB is concave up while the arc AC is concave down.

On the contrary, if the derivative $f'(x)$ decreases the slope of the tangent decreases. The tangent turns as if it "pressed down" the curve preventing it from lifting above the tangent. Therefore the arc must lie below its every tangent and thus is concave down.

Sufficient condition for concavity

Now we can take advantage of the basic theorem establishing the connection between the character of variation of a function and the sign of its derivative; as a function under consideration we take $f'(x)$ whose derivative is $(f'(x))' = f''(x)$. If $f''(x) > 0$ the derivative $f'(x)$ increases and $f''(x) < 0$ it decreases. We thus arrive at the following theorem.

Theorem If the second derivative $f''(x)$ is everywhere positive within an interval the arc of the curve $y = f(x)$ corresponding to that interval is concave up. If the second derivative $f''(x)$ is everywhere negative in an interval, the corresponding arc of the curve $y = f(x)$ is concave down.

The student should know the following mnemonic rule (the "**rain rule**"): if the graph of the function on an interval is concave up, then $y'' > 0$; if the graph of the function is concave down, then $y'' < 0$. Writing these inequality in the form $y''\underset{0}{\vee}$, $y''\underset{0}{\wedge}$ we note that the signs of the inequalities correspond to the directions of concavity of the curve ($\vee$ upward, that is , "holds water", and $\wedge$ downward, that is, "spills water").

In an interval where $f''(x)$ is positive, the function $f'(x)$ is increasing, and so the function f is concave upward. However, if a function is concave upward $f''(x)$ is not necessarily positive. For instance, $y = x^4$ is concave upward over any interval, since the derivative $4x^3$ is increasing. The second derivative $12x^2$ is not always positive; at $x = 0$ it is 0.

It should be noted that if the second derivative $f''(x)$ has constant sign, for instance, if it is positive, everywhere except at some separate points where it vanishes, the function $f'(x)$ remains increasing and the corresponding arc of the graph of the function $y = f(x)$ is concave up.

In our foregoing investigation we supposed that the function $f(x)$ in question was twice differentiable throughout the interval in question. If this condition is violated it is necessary to investigate $f'(x)$ and $f''(x)$ in the vicinity of those separate point at which the derivatives donot exist. At these points also the concavity of the graph of the function may change. An example of this kind is the graph of the function $y = \sqrt[3]{x}$, for which the point (0, 0) is a point of inflection.

Hyper-critical point

In general, a function is concave up and concave down in different parts of its domain. Suppose a function f defined in (a, b), is concave up in (a, c) and then concave down in (c, b), we are now interested in knowing what must have happened at x = c and how do we get c ? To answer these questions, we should first define the term 'hyper-critical points' or critical points of the second kind or second-order critical points.

A hyper-critical point of a function f is a number c in the domain of f such the either $f''(c) = 0$ or $f''(c)$ does not exist.

Steps for finding intervals of concavity

1. Compute the second derivative $f'(x)$ of a given function f(x), and then find the hyper-critical points i.e. points at which $f''(x)$ equals zero or does not exist .

2. Using the hyper-critical points, separate the domain of definition of the function f(x) into several intervals on each of which the derivative $f''(x)$ retains its sign. These intervals will be the intervals of concavity.

3. Investigate the sign of $f''(x)$ on each of the found intervals. If on a certain interval $f''(x) > 0$, then the function f(x) is concave up on this interval, and if $f''(x) < 0$, then f(x) is concave down on this interval.

Example 1. Find the intervals of concavity of the graph of the function $y = x^5 + 5x - 6$.

Solution We have $f'(x) = 5x^4 + 5$, $f''(x) = 20x^3$.

$f''(x)$ exists for all x. Now $f''(x) = 0$ at $x = 0$.

Hence, x = 0 is a hyper-critical point of the function.

If $x < 0$, then $f''(x) < 0$ and the curve is concave down and if $x > 0$, then $f''(x) > 0$ and the curve is concave up. Thus we see that the curve is concave down on the interval $(-\infty, 0)$ and concave up on the interval $(0, \infty)$.

Example 2. Find the intervals in which the curve $y = x \sin(\ln x)$, $x > 0$ is concave up or concave down.

Solution We find the derivatives :

$$y' = \sin(\ln x) + \cos(\ln x),$$

$$y'' = \frac{1}{x}[\cos(\ln x) - \sin(\ln x)] = \frac{\sqrt{2}}{x}\sin\left(\frac{\pi}{4} - \ln x\right).$$

The second derivative vanishes at the points

$$x_k = e^{\pi/4 + k\pi}, k = 0, \pm 1, \pm 2, ...$$

The function $\sin(\pi/4 - \ln x)$, and together with it y'', changes sign when passing through each point x_k.

In the intervals $(e^{2k\pi - 3\pi/4}, e^{2k\pi + \pi/4})$ the curve is concave up, and in the intervals $(e^{2k\pi + \pi/4}, e^{2k\pi + 5\pi/4})$ it is concave down.

Method for finding Point of Inflection

An important characteristic of a curve are the points separating its concave up and concave down arcs.

Suppose the graph of a function f has a tangent line (possibly vertical) at the point P(c, f(c)) and that the graph is concave up on one side of P and concave down on the other side. Then P is called an inflection point of the graph.

The concavity of the graph of f will change only at points where $f''(x) = 0$ or $f''(x)$ does not exist–that is, at the hyper-critical points.

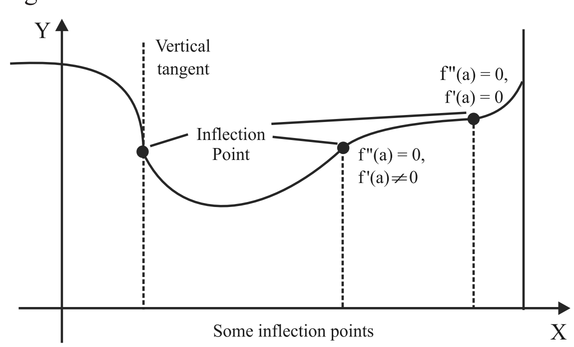
If x = c is a hyper-critical point and the inequalities $f''(c-h) < 0, f''(c+h) > 0$ (or inequalities $f''(c-h) > 0$, $f''(c+h) < 0$) hold for an arbitrary sufficiently small $h > 0$, then the point of the curve y = f(x) with the abscissa x = c is a point of inflection.

If $f''(c-h)$ and $f''(c+h)$ are of the same sign, then the point x = c is not a point of inflection.

If the second derivative $f''(x)$ changes sign as x passes through x = c (from the left to right) then c is a point of inflection, if it changes sign from – to + there is an interval of concave down on the left of the point c and interval of concave up on the right of it, and, conversely, if it changes sign from + to –, an interval of concave down follows an interval of concave up as x passes through c.

For example, the second derivative of the function $y = x^5 + x$ is $y'' = 20x^3$, and $y''(0) = 0$.

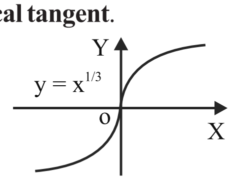
We have $y'' < 0$ for $x < 0$ and $y'' > 0$ for $x > 0$. Consequently, the origin is a point of inflection of the graph of the given function, the interval of concave down lying on the left of it and the interval of concave up on the right.



Some inflection points

Note:

1. A continuous function f need not have an inflection point where $f''(x) = 0$. For instance, if $f(x) = x^4$, we have $f''(0) = 0$, but the graph of f is always concave up.
   Let us take the function $y = x^5 + 5x^4$. Here $y'' = 20x^2(x+3)$, and $y'' = 0$ for $x = -3$ and for $x = 0$. As x passes through the point $x = -3$, the second derivative changes sign, and thus $x = -3$ is a point of inflection.
   When x passes through the point $x = 0$, the second derivative retains constant sign, and therefore, the origin is not a point of inflection; the graph of the given function is concave up on both sides of the origin.
2. If $x = c$ is a point of inflection of a curve $y = f(x)$ and at this point there exists the second derivative $f''(c)$, then $f''(c)$ is necessarily equal to zero ($f''(c) = 0$).
3. The point (–1, 0) in $y = (x - 1)^3$, being both a critical point and a point of inflection, is a point of **horizontal inflection**.
4. If a function f is such that the derivative f''' is continuous at $x = c$ and $f''(c) = 0$ while $f'''(c) \neq 0$, then the curve $y = f(x)$ has a point of inflection for $x = c$.
5. It should be noted that a point separating a concave up arc of a curve from a concave down one may be such that the tangent at that point is perpendicular to the x-axis i.e. a vertical tangent or such that the tangent does not exist.
   This is demonstrated by the behaviour of the graph of the function $y = \sqrt[3]{x}$ in the vicinity of the origin. In such a case we speak of a **point of inflection with vertical tangent**.



6. A number c such that $f''(c)$ is not defined and the concavity of f changes at c will correspond to an inflection point if and only if f(c) is defined.

Example 3. Find the inflection points of the graph of the function $f(x) = x^4 - 4x^3 + x - 7$.

Solution Since $f''(x) = 12x^2 - 24x = 12x(x-2)$, we have $f''(x) = 0 \Rightarrow x = 0$ or $x = 2$.

Thus, the points (0, –7), (2, –21) are the only possible inflection points.

Now since $12x(x - 2) < 0$ for $0 < x < 2$ and $12x(x-2) > 0$ for $x < 0$ it is clear that (0, –7) is in fact an inflection point.

Similarly, since $12x(x-2) > 0$ for $x > 2$ and $12x(x-2) < 0$ for $0 < x < 2$, the change in sign guarantees that (2, –21) also is an inflection point.

Example 4. Find the points of inflection of the function $f(x) = \sin^2 x$, $x \in [0, 2\pi]$.

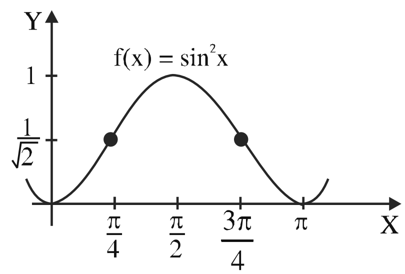
Solution $f(x) = \sin^2 x$

$f'(x) = \sin 2x$

$f''(x) = 2\cos 2x$

$$f''(0) = 0 \Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}$$

Both these points are inflection points as sign of $f''(x)$ change about these points.



Example 5. Find the inflection points of the curve $y = (x-5)^{5/3} + 2$.

Solution We find $y' = \frac{5}{3}(x-5)^{2/3}$,

$$y'' = \frac{10}{9\sqrt[3]{(x-5)}}.$$

The second derivative does not vanish for any value of x and does not exist at $x = 5$.

The tangent exists at $x = 5$.

Since $y''(5-h) < 0$, $y''(5+h) > 0$, the point $x = 5$ is the abscissa of the inflection point Thus, (5, 2) is the inflection point.

Example 6. Find the points of inflection of the graph of the function $y = \sqrt[3]{x+2}$.

Solution We have

$$y'' = -\frac{2}{9}(x+2)^{-\frac{5}{3}} = \frac{-2}{9\sqrt[3]{(x+2)^5}} \quad ...(1)$$

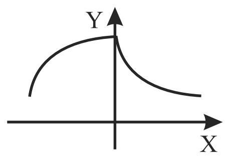
It is obvious that y" does not vanish anywhere.

Equating to zero the denominator of the fraction on the right of (1), we find that y" does not exist for $x = -2$.

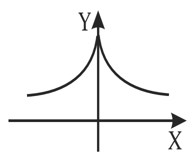
The tangent at this point is parallel to the y-axis, since the first derivative y' is infinite at $x = -2$.

Since $y'' > 0$ for $x < -2$ and $y'' < 0$ for $x > -2$, it follows that $(-2, 0)$ is the point of inflection.

Note: We can draw a curve with a corner point separating its concave up and concave down arcs.



We shall not include corner points of this kind into the class of points of inflection. There may of course exist a corner point at which the character of concavity of the curve does not change. See the figure below.



Cusp

If $f'(x)$ approaches ∞ from one side of a point $x = c$ and $-\infty$ from the other side, then the function f is said to have a **cusp** at $x = c$.

Let $f(x) = 2x^{5/3} + 5x^{2/3}$.

$$f'(x) = 2\left(\frac{5}{3}\right) x^{2/3} + 5\left(\frac{2}{3}\right) x^{-1/3} = \frac{10}{3} x^{-1/3} (x+1)$$

$$f''(x) = \frac{10}{3}\left(\frac{2}{3}\right)x^{-1/3} + \frac{10}{3}\left(-\frac{1}{3}\right) x^{-4/3}$$

$$= \frac{10}{9} x^{-4/3} (2x-1)$$

Note that the graph is concave down on both sides of $x = 0$ and that the slope $f'(x)$ decreases without bound to the left of $x = 0$ and increases without bound to the right. This means the graph changes direction abruptly at $x = 0$, and we have a cusp at the origin.

Example 7. Find the points of inflection of the curve $y = 2 - |x^5 - 1|$.

Solution The given function can be written as :

$$y = \begin{cases} 2 - (x^5 - 1), & x \ge 1, \\ 2 + (x^5 - 1), & x < 1. \end{cases}$$

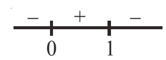
Therefore, $y' = \begin{cases} -5x^4, & x > 1, \\ 5x^4, & x < 1. \end{cases}$

At the point $x = 1$ there is no derivative.

Further, $y'' = \begin{cases} -20x^3, & x > 1, \\ 20x^3, & x < 1; \end{cases}$

$y'' = 0$ at the point $x = 0$. Hence we have to investigate three intervals : $(-\infty, 0), (0, 1), (1, \infty)$.

Sign scheme of y"



The point (0, 1) is a point of inflection, the point (1, 2) being a corner point.

Example 8. What conditions must the coefficients a, b, c satisfy for the curve $y = ax^4 + bx^3 + cx^2 + dx + e$ to have points of inflection ?

Solution Find the second derivative :

$y'' = 12ax^2 + 6bx + 2c$

The curve has points of inflection if and only if the equation $6ax^2 + 3bx + c = 0$

has different real root, i.e. when the discriminant

$9b^2 - 24ac > 0$,

or $3b^2 - 8ac > 0$.

Example 9. Consider a curve $C : y = \cos^{-1}(2x - 1)$ and a straight line $L : 2px - 4y + 2\pi - p = 0$

Find the set of values of 'p' for which the line L intersects the curve at three distinct points.

Solution $y = \cos^{-1}(2x - 1)$

$$\frac{dy}{dx} = \frac{-2}{\sqrt{1-(2x-1)^2}} = \frac{-1}{\sqrt{x-x^2}} = -(x - x^2)^{-1/2}$$

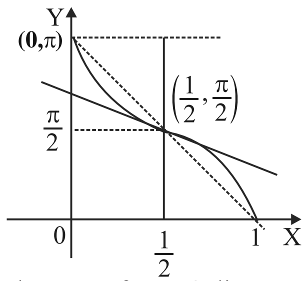
$$\frac{d^2y}{dx^2} = \frac{1}{2}\frac{(1-2x)}{(x-x^2)^{3/2}} = 0 \Rightarrow x = \frac{1}{2}$$

The point $\left(\frac{1}{2}, \frac{\pi}{2}\right)$ is a point of inflection of the curve is and it satisfies the line L.

The line L is always passing through point of inflection of the curve C.

Slope of the tangent to the curve C at $\left(\frac{1}{2}, \frac{\pi}{2}\right)$

$$\left.\frac{dy}{dx}\right|_{x=1/2} = -2$$



As the slope decreases from –2, line cuts the curve at three distinct points and minimum slope of the line when it intersects the curve at three distinct points is

$$\frac{\pi - \frac{\pi}{2}}{0 - \frac{1}{2}} = -\pi$$

$\therefore\ \frac{p}{2} \in [-\pi, -2) \Rightarrow p \in [-2\pi, -4)$

Example 10. Prove the inequality

$$\sin x + 2x \geq \frac{3x(x+1)}{\pi} \quad \forall\ x \in (0, \pi/2).$$

Solution $f(x) = \sin x + 2x$

$$g(x) = \frac{3(x^2 + x)}{\pi}$$

$f'(x) = \cos x + 2 > 0$; $f''(x) = -\sin x < 0$

Hence f is concave down and increasing.

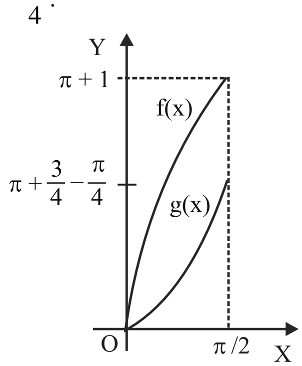
$$g'(x) = \frac{3}{\pi}(2x+1) \qquad g''(x) = \frac{3}{\pi}(2) > 0$$

$\Rightarrow$ g is concave up and increasing.

$$f\left(\frac{\pi}{2}\right) = \pi + 1$$

$$g\left(\frac{\pi}{2}\right) = 3\frac{\pi}{2\pi}\left(\frac{\pi}{2}+1\right) = \frac{3}{2}\left(\frac{\pi}{2}+1\right) = \frac{3\pi}{4} + \frac{3}{2}$$

$$= \pi + \frac{3}{4} - \frac{\pi}{4}.$$



From the graph, it is clear that $f(x) \geq g(x)\ \forall\ x \in (0, \pi/2)$.

Example 11. Find equations of the tangent lines at the points of inflection of $y = f(x) = x^4 - 6x^3 + 12x^2 - 8x$.

Solution $f(x) = 4x^3 - 18x^3 + 24x - 8$

$f''(x) = 12x^2 - 36x + 24 = 12(x-1)(x-2)$

$f'''(x) = 24x - 36 = 12(2x - 3)$

The possible points of inflection are at x = 1 and 2. Since $f'''(1) \neq 0$ and $f'''(2) \neq 0$, the points (1, – 1) and (2,0) are points of inflection.

At (1, – 1), the slope of the tangent line is

$m = f'(1) = 2$, and its equation is

$y = y_1 = m(x - x_1)$ or $y + 1 = 2(x - 1)$

or $y = 2x - 3$

At (2, 0), the slope is $f'(2) = 0$, and the equation of the tangent line is y = 0.

Example 12. Let $f'(x) > 0$ and $f''(x) > 0$ where $x_1 < x_2$. Then show that $f\left(\frac{x_1 + x_2}{2}\right) < \frac{f(x_1) + f(x_2)}{2}$.

Solution Since $f(x) > 0$ and $f''(x) > 0$, the function is strictly increasing and concave up. A sample graph of f has been shown in the figure below.

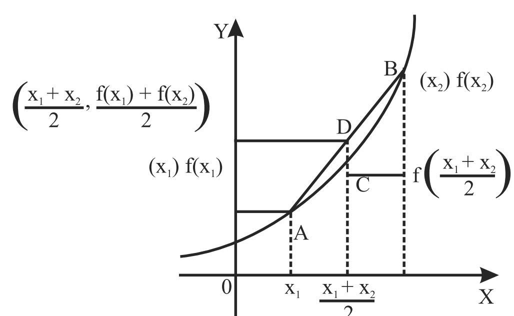
We know, $x_1 < \frac{x_1 + x_2}{2} < x_2$

and $\left(\frac{x_1 + x_2}{2}, \frac{f(x_1) + f(x_2)}{2}\right)$ is the midpoint of the chord joining A and B.

Since the graph is concave up

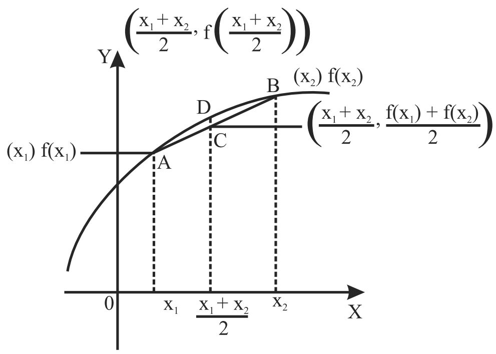
$$f\left(\frac{x_1 + x_2}{2}\right) < \frac{f(x_1) + f(x_2)}{2}$$

because ordinate of a point on the curve is less than that of a point on the chord at the same abscissa.



Similarly, for a concave down graph as shown in the figure below, we have

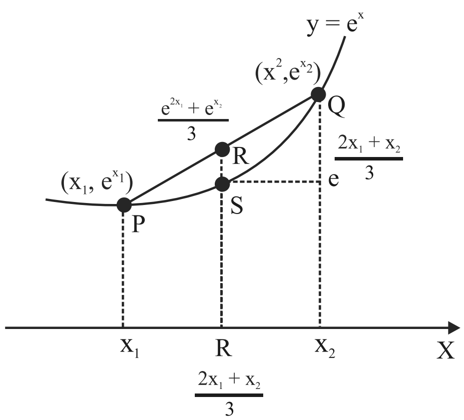
$$f\left(\frac{x_1+x_2}{2}\right) > \frac{f(x_1)+f(x_2)}{2}$$



Example 13. Prove that for any two numbers x_1 and x_2,

$$\frac{2e^{x_1}+e^{x_2}}{3} > e^{\frac{2x_1+x_2}{3}}$$

Solution Assume $f(x) = e^x$ and let x_1 and x_2 be two points on the curve $y = e^x$.



Let R be another point which divides P and Q in ratio 1 : 2. The y coordinate of point R is $\frac{2e^{x_1}+e^{x_2}}{3}$ and y coordinate of point S is $e^{\frac{2x_1+x_2}{3}}$.

Since $f(x) = e^x$ is always concave up, hence point R will always be above point S.

$$\Rightarrow \quad \frac{2e^{x_1}+e^{x_2}}{3} < e^{\frac{2x_1+x_2}{3}}$$

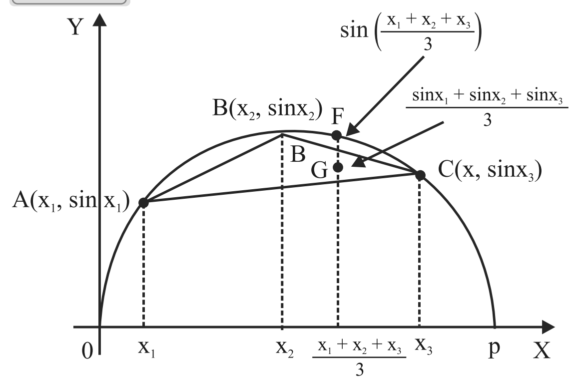
Example 14. If $0 < x_1 < x_2 < x_3 < \pi$ then prove that

$$\sin\left(\frac{x_1+x_2+x_3}{3}\right) > \frac{\sin x_1+\sin x_2+\sin x_3}{3}.$$

Hence or otherwise prove that if A, B, C are angles of triangle then maximum value of

$$\sin A + \sin B + \sin C \text{ is } \frac{3\sqrt{3}}{2}.$$

Solution



Let point A, B, C form a triangle y coordinate of centroid G is $\frac{\sin x_1+\sin x_2+\sin x_3}{3}$ and y coordinate of point F is $\sin\left(\frac{x_1+x_2+x_3}{3}\right)$.

Hence $\sin\left(\frac{x_1+x_2+x_3}{3}\right) > \frac{\sin x_1+\sin x_2+\sin x_3}{3}$.

If $A+B+C=\pi$, then

$$\sin\left(\frac{A+B+C}{3}\right) > \frac{\sin A+\sin B+\sin C}{3}$$

$$\Rightarrow \quad \sin\frac{\pi}{3} > \frac{\sin A+\sin B+\sin C}{3}$$

$$\Rightarrow \quad \frac{3\sqrt{3}}{2} > \sin A + \sin B + \sin C$$

$\Rightarrow$ maximum value of $(\sin A+\sin B+\sin C)$

$= \frac{3\sqrt{3}}{2}$. It is attained if $A = B = C$

Note:

Let $g(x) = f^{-1}(x)$

Since g is the inverse of f, $f \circ g(x) = g \circ f(x) = x$

$\Rightarrow \quad g'(x) = \frac{1}{f'(g(x))}$

$\Rightarrow \quad g'(x) > 0$ {as f(x) is increasing} ...(1)

$\Rightarrow \quad g(x)$ is increasing for all $x \in R$.

$\Rightarrow \quad f^{-1}(x)$ is increasing for all $x \in R$.

again $\quad g'(x) = \frac{1}{f'(g(x))}$

$\Rightarrow$ $g''(x) = -\dfrac{1}{(f'(g(x))^2} f''(g)\, g'(x)$, for all $x \in R$

$\Rightarrow$ $g''(x) < 0$, $\begin{cases} \text{as } g'(x) > 0, \text{ from(1)} \\ f''(x) > 0, \text{ given} \end{cases}$

$\Rightarrow$ $g'(x)$ is decreasing for all $x \in R$.

$\Rightarrow$ $f^{-1}(x)$ is increasing and $\dfrac{d}{dx}\{f^{-1}(x)\}$ is decreasing.

This means that if a function f is strictly increasing and concave up, then its inverse f^{-1} is strictly increasing and concave down.

Concept Problems E

1. Let f be a function such that $f''(x) = (x-1)(x-2)$.
   (i) For which x is f concave upward?
   (ii) For which x is f concave downward?
   (iii) List its inflection points.
2. Show that the graph of the function $y = x \tan^{-1} x$ is concave up everywhere.
3. Is it true that the concavity of the graph of a twice differentiable function $y = f(x)$ changes every time $f''(x) = 0$? Give reasons for your answer.
4. Show that the graph of the quadratic function $y = Ax^2 + Bx + C$ is concave up if $A > 0$ and concave down if $A < 0$.
5. Let $f(x) = ax^2 + bx + c$, where a, b and c are constants, $a \neq 0$. Show that f has no inflection points.
6. Explain why a polynomial of odd degree (atleast 3) always has atleast one inflection point.
7. Let $f(x) = x^{2/3}|x|$. Is (0, 0) a point of inflection of this graph? Show that$f''(0)$ does not exist.
8. Graph the function $y = x^{1/3}$ and show that it has a point of inflection where neither the first nor the second derivative exists.
9. Sketch the graph of $f(x) = \sqrt[5]{x^3}$ and identify the inflection point. Does $f''(x)$ exist at the inflection point?
10. Show that the function $g(x) = x|x|$ has an inflection point at (0, 0) but g"(0) does not exist.
11. Prove (without referring to a picture) that if the graph of f lies above its tangent lines for all x in [a,b], then $f''(x) \ge 0$ for all x in [a,b].
12. If f is a function such that $f''(x) > 0$ for all x, then prove that the graph of $y = f(x)$ lies below its chords; i.e., $f(ax_1 + (1-a)x_2) < af(x_1) + (1-a) f(x_2)$ for any a in (0,1), and for any x_1 and x_2, $x_1 \neq x_2$.
13. Assume that all of the functions are twice differentiable and the second derivatives are never 0 on an interval I.
    (a) If f and g are concave up on I, show that f + g is concave up on I.
    (b) If f is positive and concave up on I, show that the function $g(x) = [f(x)]^2$ is concave up on I.
    (c) If f and g are positive, increasing, concave up functions on I, show that the product function fg is concave up on I
    (d) Show that part (iii) remains true if f and g are both decreasing.
    (e) Suppose f is increasing and g is decreasing. show, by giving examples, that fg may be concave up, concave down, or linear. Why doesn't the argument in parts (iii) and (iv) work in this case ?
14. For any twice derivable function f on (a, b), prove that the function $g(x) = f(x) + cx + d$, where c and d are any real numbers, has the same concavity characteristics as f.
15. If $f(x) > 0$ and $g(x) > 0$ for all x on I and if f and g are concave up on I, then is fg also concave up on I ?
16. Suppose $f(x) > 0$ on (a, b) and ln f is concave up. Prove that f is concave up.

Practice Problems E

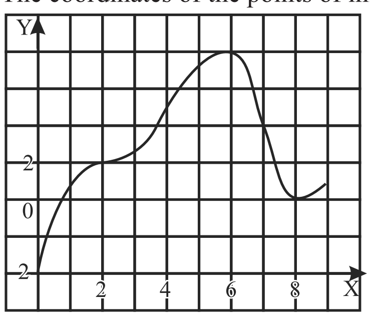
17. Find out whether the curve $y = x^4 - 5x^3 - 15x^2 + 30$ is concave up or concave down in the vicinity of the points (1, 11) and (3, 3).
18. Prove that the curve $y = x.\ln x$ is everywhere concave up.
19. Find the points of inflection of the curve $y = \dfrac{x^2}{a^2 + x^2}$.
20. Determine the values of a, b and c if the graph of $f(x) = ax^3 + bx^2 + c$ is to have (–1, 1) as a point of inflection of f at which the slope is 2.

21. An inflection point of a graph is called a horizontal inflection point if the slope there is zero. Find the horizontal inflection points of
(i) x^3, (ii) $(x-a)^3$,
(iii) $(x-a)^3 + b$.

22. Find the ranges of values of x in which the curves
(i) $y = 3x^5 - 40x^3 + 3x - 20$
(ii) $y = (x^2 + 4x + 5)e^{-x}$
are concave up or concave down. Also find their points of inflection.

23. Find the intervals in which the curve $y = (\cos x + \sin x)\, e^x$ is concave up or down for $x \in (0, 2\pi)$.

24. Use the given graph of f to find the following.
(i) The intervals on which f is increasing.
(ii) The intervals on which f is decreasing.
(iii) The intervals on which f is concave up.
(iv) The intervals on which f is concave down.
(v) The coordinates of the points of inflection.



25. Find out whether the curve $y = x^2 \ln x$ is concave up or concave down in the neighbourhoods of the points (1, 0) and $\left(\frac{1}{e^2}, -\frac{2}{e^4}\right)$.

26. Find the equations of the tangent lines at all inflection points on the graph of
$f(x) = x^4 - 6x^3 + 12x^2 - 8x + 3$

27. Find the point of inflection of the graph of
$y = x^2 - \dfrac{1}{6x^3}$
Find the equation of the tangent line to the graph at this point.

28. Show that the curve $y = \dfrac{1+x}{1+x^2}$ has three points of inflection, and that they lie in a straight line.

29. Consider the function $f(x) = \dfrac{\cos x}{x}$.
If $0 < x_1 < x_2 < \dfrac{\pi}{2}$ consider two expressions

$$\frac{\cos x_1}{x_1} + \frac{\cos x_2}{x_2} \text{ and } \frac{\cos\left(\frac{x_1+x_2}{2}\right)}{x_1+x_2}.$$

Prove that $\dfrac{\cos x_1}{x_1} + \dfrac{\cos x_2}{x_2} > \dfrac{\cos\left(\frac{x_1+x_2}{2}\right)}{x_1+x_2}$.

30. Show that the curve $y = e^{-x^2}$ has inflection at the points for which $x = \pm \dfrac{1}{\sqrt{2}}$.

Target Exercises for JEE Advanced

Problem 1. Prove that the function $f(x) = x^2 \sin(1/x) + ax$, where $0 < a < 1$, when $x \neq 0$, and $f(0) = 0$ is not monotonic in any interval containing the origin.

Solution We have $f'(0) = a > 0$. (using first principles)
Thus the conditions of the theorem for increasing functions are satisfied.
Hence, f is strictly increasing at $x = 0$.
But for $x \neq 0$, $f'(x) = 2x\sin(1/x) - \cos(1/x) + a$
which oscillates between the limits $a - 1$ and $a + 1$ as $x \to 0$.
Since $a - 1 < 0$, we can find values of x, as near to 0 as we like, for which $f'(x) < 0$; and it is therefore impossible to find any interval, including $x = 0$, througout which f(x) is a strictly increasing function .

Problem 2. Let f be the function defined by
$f(x) = (ax^2 + 2hx + b) / (px^2 + 2qx + r)$,
for all x for which the denominator does not vanish. Prove that the stationary values of f are the roots of
$(pr - q^2)\lambda^2 - (pb + ra - 2qh)\lambda + (ab - h^2) = 0$

Solution Since $f(x) = (ax^2 + 2hx + b)/(px^2 + 2qx + r)$,

$$f'(x) = \frac{2\{(ax+h)(px^2+2qx+r) - (px+q)(ax^2+2hx+b)\}}{(px^2+2qx+r)^2}$$

If f has a stationary point at $x = x_0$, then $f'(x_0) = 0$,
i.e., $(ax_0 + h)(px_0^2 + qx_0 + r) - (px_0 + q)(ax_0^2 + 2hx_0 + b) = 0$ ...(1)
Also, the value λ of f at $x = x_0$ is given by
$\lambda = (ax_0^2 + 2hx_0 + b)/(px_0^2 + 2qx_0 + r)$, ...(2)

From (1) and (2), we have

$(ax_0 + h) - \lambda(px_0 + q) = 0$ ...(3)

Also, we may re-write (1) as

$(px_0^2 + 2qx_0 + r)/(px_0 + q) = (ax_0^2 + 2hx_0 + b)/(ax_0 + h)$,

i.e. $(qx_0 + r)/(px_0 + q) = (hx_0 + b)/(ax_0 + h)$,

i.e. $(hx_0 + b) - \lambda(qx_0 + r) = 0$, ...(4)

by using (3),

Eliminating x_0 from (3) and (4), we have

$$\begin{vmatrix} a - \lambda p & h - \lambda q \\ h - \lambda q & b - \lambda r \end{vmatrix} = 0$$

i.e. $(pr - q^2)\lambda^2 - (pb + ra - 2qh)\lambda + ab - h^2 = 0$.

Problem 3. Find the intervals of monotonicity of the function $y = 2x^2 - \ln|x|, x \neq 0$.

Solution Let $f(x) = 2x^2 - \ln|x|$ ...(1)

Case I : $x < 0$.

$f(x) = 2x^2 - \ln|x| = 2x^2 - \ln(-x)$

$\therefore\ f'(x) = 4x - \dfrac{1}{(-x)}(-1) = 4x - \dfrac{1}{x} = \dfrac{4x^2 - 1}{x}$ ...(2)

Case II : $x > 0$.

$f(x) = 2x^2 - \ln|x| = 2x^2 - \ln x$

$\therefore\ f'(x) = 4x - \dfrac{1}{x}$ ...(3)

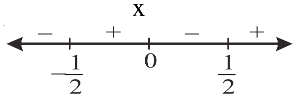
Thus when $x < 0$ or $x > 0$,

$f'(x) = 4x - \dfrac{1}{x} = \dfrac{4x^2 - 1}{x}$ ...(4)

Sign scheme for $f'(x)$ i.e. $\dfrac{4x^2 - 1}{x}$:

when $4x^2 - 1 = 0$, $x = \pm\dfrac{1}{2}$

$\therefore$ Sign scheme for $\dfrac{4x^2 - 1}{x}$ is as follows



Thus, f(x) will be a decreasing function in the interval

$\left(-\infty, -\dfrac{1}{2}\right)$ and $\left(0, \dfrac{1}{2}\right)$

and increasing function in $\left(-\dfrac{1}{2}, 0\right)$ and $\left(0, \dfrac{1}{2}\right)$.

Problem 4. If $f(x) = \{-c^2 + (b-1)c - 2\}x + \displaystyle\int_0^x (\sin^2 x + \cos^4 x)dx$ is an increasing function $\forall\ x \in R$ then find all possible values of b, for any $c \in R$.

Solution Given

$$f(x) = \{-c^2 + (b-1)c - \}x + \int_0^x (\sin^2 x + \cos^4 x)dx$$

$\therefore\ f'(x) = -c^2 + (b-1)c - 2 + \sin^2 x + \cos^4 x$

$\Rightarrow\ f'(x) = (1 - \sin^2 x)^2 + \sin^2 x - c^2 + (b-1)c - 2$

$= \sin^4 x - \sin^2 x - c^2 + (b-1)c - 1$

$= \left(\sin^2 x - \dfrac{1}{2}\right)^2 - c^2 + (b-1)c - \dfrac{5}{4}$

We should have $f'(x) \geq 0\ \forall\ x \in R$

$\therefore\ \left(\sin^2 x - \dfrac{1}{2}\right)^2 - c^2 + (b-1)c - \dfrac{5}{4} \geq 0$

$0 - c^2 + (b-1)c - \dfrac{5}{4} \geq 0$

(since the least value of $(\sin^2 x - 1/2)^2 = 0$)

or $c^2 - (b-1)c + \dfrac{5}{4} \leq 0$ for all c.

$\therefore\ D \leq 0 \Rightarrow (b-1)^2 \leq 5$

$\Rightarrow\ -\sqrt{5} \leq b - 1 \leq -\sqrt{5}$

$\Rightarrow\ 1 - \sqrt{5} \leq b \leq 1 + \sqrt{5}$

Hence $b \in [1 - \sqrt{5}, 1 + \sqrt{5}]$.

Problem 5. Find the intervals of monotonicity of the function $f(x) = x\sqrt{ax - x^2}$, $a \neq 0$

Solution $y = x\sqrt{ax - x^2}$

$\dfrac{dy}{dx} = \sqrt{ax - x^2} + \dfrac{x}{2}\dfrac{(a - 2x)}{\sqrt{ax - x^2}}$

$\Rightarrow\ \dfrac{dy}{dx} = \dfrac{x}{2}\dfrac{(3a - 4x)}{2\sqrt{ax - x^2}}$

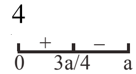
Domain : $ax - x^2 \geq 0 \Rightarrow x(x - a) \leq 0$

Case I If $a > 0$, then $x \in [0, a]$

Case II If $a < 0$, then $x \in [a, 0]$

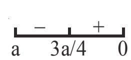
Critical point : $x = \dfrac{3a}{4}$

Case I : $a > 0$, Sign of $f'(x)$



f(x) is increasing in $(0, \dfrac{3a}{4})$ and decreasing in $(\dfrac{3a}{4}, a)$

Case II : $a < 0$, Sign of $f'(x)$



f(x) is decreasing in $(a, \dfrac{3a}{4})$ and increasing in $(\dfrac{3a}{4}, 0)$.

Problem 6. Find the intervals of monotonicity of the function $f(x) = \frac{a}{x}\ln\frac{x}{a}$.

Solution $y = \frac{a}{x}\ln\frac{x}{a}$

$$\frac{dy}{dx} = \frac{a}{x^2}\ln\frac{x}{a} + \frac{x}{a}\frac{1}{x/a}\cdot\frac{1}{a}$$

$$\frac{dy}{dx} = -\frac{a}{x^2}\ln\frac{x}{a} + \frac{a}{x^2} = \frac{a}{x^2}[1 - \ln x/a]$$

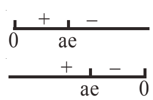
$$\text{Domain}: \begin{cases} x > 0 & \text{if } a > 0 \\ x < 0 & \text{if } a < 0 \end{cases}$$

Critical point : $\ln\left(\frac{x}{a}\right) = 1 \Rightarrow x = ae$

Sign of $f'(x)$

$a > 0$

$a < 0$



If $a > 0$, $f(x)$ is increasing in $(0, ae)$ and decreasing in (ae, ∞). If $a < 0$, $f(x)$ is increasing in $(-\infty, ae)$ and decreasing in $(ae, 0)$.

Problem 7. Find the number of real roots of the equation $\sum_{i=1}^{n}\frac{a_i^2}{x - b_i} = c$, where $b_1 < b_2 < \ldots < b_n$.

Solution Consider the function

$$f(x) = \sum_{i=1}^{n}\frac{a_i^2}{x - b_i} - c = \frac{a_1^2}{x - b_1} + \frac{a_2^2}{x - b_2} + \ldots + \frac{a_n^2}{x - b_n} - c$$

$$\text{and } f'(x) = \left[\frac{a_1^2}{(x - b_1)^2} + \frac{a_2^2}{(x - b_2)^2} + \ldots + \frac{a_n^2}{(x - b_n)^2}\right]$$

$$< 0 \ \forall\, x \in R - \{b_1, b_2, \ldots, b_n\}$$

$\Rightarrow$ $f(x)$ strictly decreases in $(-\infty, b_1)$, (b_1, b_2), ..., (b_{n-1}, b_n)

Now, we have

$f(-\infty) = -c = f(\infty)$

$f(b_1^-) = -\infty$ and $f(b_1^+) = \infty$

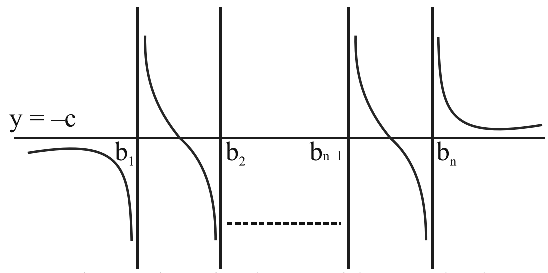
$f(b_2^-) = -\infty$ and $f(b_2^+) = \infty$

....

....

$f(b_n^-) = -\infty$ and $f(b_n^+) = \infty$

The plot of the curve $y = f(x)$ is shown alongside.



Hence, the number of real roots of the equation is $n - 1$.

Problem 8. If $f: R \to R$ and f is a polynomial with $f(x) = 0$ has real and distinct roots, show that the equation, $[f'(x)]^2 - f(x) . f''(x) = 0$ cannot have real roots.

Solution Let $f(x) = c(x - x_1)(x - x_2) \ldots\ldots (x - x_n)$

Again Let $h(x) = \frac{f'(x)}{f(x)}$

$$= \left(\frac{1}{x - x_1} + \frac{1}{x - x_2} + \ldots\ldots + \frac{1}{x - x_n}\right)$$

$$h'(x) = \frac{f(x) . f''(x) - [f'(x)]^2}{f^2(x)}$$

$$= -\left(\frac{1}{(x - x_1)^2} + \frac{1}{(x - x_2)^2} + \ldots\ldots + \frac{1}{(x - x_n)^2}\right)$$

$\Rightarrow$ $h'(x) < 0 \Rightarrow f(x) . f''(x) - [f'(x)]^2 < 0$

Alternatively, a function $f(x)$ satisfying the equation $[f'(x)]^2 - f(x) . f''(x) = 0$ is

$f(x) = c . e^{c_1 x}$ which cannot have any root.

Problem 9. Consider the function,

$f(x) = x^3 - 9x^2 + 15x + 6$ for $1 \le x \le 6$ and

$$g(x) = \begin{cases} \min. f(t) & \text{for } 1 \le t \le x, 1 \le x \le 6 \\ x - 18 & \text{for } x > 6 \end{cases}$$

then prove that

(i) $g(x)$ is differentiable at $x = 1$ (ii) $g(x)$ is discontinuous at $x = 6$ (iii) $g(x)$ is continuous and derivable at $x = 5$ (iv) $g(x)$ is monotonic in $(1, 5)$

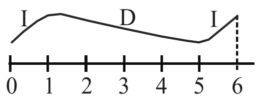
Solution $f(x) = x^3 - 9x^2 + 15x + 6$

$f(t) = t^3 - 9t^2 + 15t + 6$

$f'(t) = 3t^2 - 18t + 15 = 3[t^2 - 6t + 5]$

$= 3(t - 5)(t - 1)$

Hence f is increasing in $(5, 6)$ and f is decreasing in $(1, 5)$



Now $g(x)=\begin{cases} f(x)=x^3-9x^2+15x+6 & 1\le x<5 \\ f(5)=-19 & 5\le x\le 6 \\ x-18 & x>6 \end{cases}$

$\therefore\ g(x)=\begin{cases} x^3-9x^2+15x+6 & 1\le x<5 \\ -19 & 5\le x\le 6 \\ x-18 & x>6 \end{cases}$

Hence, g is continuous and differentiable at $x = 1$
g is continuous and differentiable at $x = 5$
g is neither continuous nor derivable at $x = 6$
$g(x)$ is monotonic in $(1, 5)$.

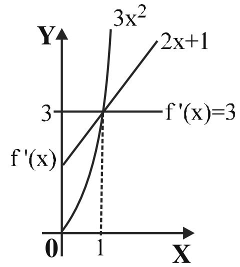
Problem 10. Given $f : [0, \infty) \longrightarrow R$ be a strictly increasing function such that the functions $g(x) = f(x) - 3x$ and $h(x) = f(x) - x^3$ are both strictly increasing function. Then prove that the function $F(x) = f(x) - x^2 - x$ is increasing throughout $(0, \infty)$.

Solution $3F(x)=3[f(x)-x^2-x]$

$$=\underbrace{2[f(x)-3x]}_{g(x)}+\underbrace{\left(f(x)-x^3\right)}_{h(x)}+x^3-3x^2+3x-1+1$$

$3F(x)=2g(x)+h(x)+(x-1)^3+1$
$\Rightarrow$ $F(x)$ is increasing $\forall\ x\in[0,\infty]$.

***Alternative*:**
Given $f(x)$ is increasing
$\Rightarrow\ f'(x)>0$
$g(x)=f(x)-3x$
$g'(x)=f'(x)-3>0 \Rightarrow f'(x)>3$
also $h(x)=f(x)-x^3$
$h'(x)=f'(x)-3x^2>0 \Rightarrow f'(x)>3x^2$
To prove that $F(x)=f(x)-x^2-x$ is increasing
i.e $F'(x)=f'(x)-2x-1>0$
$F'(x)>0$
$f'(x)>2x+1$



For in $[0, 1)$, obviously $f'(x)>3>2x+1$ and in $(1, 2)$, $3x^2>2x+1$. Hence proved.

Problem 11. Let $f:(0,\infty)\longrightarrow(0,\infty)$ be a derivable function and $F(x)$ is the antiderivative of $f(x)$ such that $2\left(F(x)-f(x)\right)=f^2(x)$ for any real positive x. Then prove that f is strictly increasing and $\lim\limits_{x\to\infty}\frac{f(x)}{x}=1$.

Solution Given $2(F(x)-f(x))=f^2(x)$

and $\frac{dF}{dx}=f(x)$

$\therefore\ F(x)=\frac{f^2(x)}{2}+f(x)$
$F'(x)=f(x)\cdot f'(x)+f'(x)$
$\therefore\ f(x)=f'(x)(1+f(x))$

$\therefore\ f'(x)=\frac{f(x)}{1+f(x)}=1-\frac{1}{1+f(x)}>0$ (as $f(x)>0$)

Hence f is strictly increasing.

$$\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}f'(x)\quad \text{using L'Hospital's Rule}$$

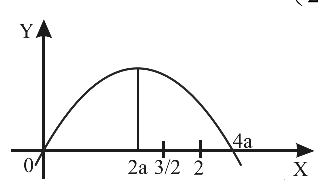
$$=\lim_{x\to\infty}\left(1-\frac{1}{1+f(x)}\right)\qquad \text{as } x\longrightarrow\infty,\ f(x)\longrightarrow\infty$$

$\therefore\ \lim\limits_{x\to\infty}\frac{f(x)}{x}=1.$

Problem 12. Find all possible values of 'a' for which $f(x) = \log_a(4ax - x^2)$ is strictly increasing for every $x\in\left[\frac{3}{2},2\right]$.

Solution
Case I : If $0<a<1$ (obviously 'a' cannot be <0) then for $f(x)$ to be increasing

$4ax-x^2$ should be decreasing in $\left(\frac{3}{2},2\right)$



$\Rightarrow\ \frac{3}{2}\ge 2a$ and $2<4a$ $\quad\Rightarrow\ a\le\frac{3}{4}$ and $a>\frac{1}{2}$

$\Rightarrow\ a\in\left(\frac{1}{2},\frac{3}{4}\right]$

Case II : If $a > 1$ then for f (x) to be increasing

$4ax - x^2$ increasing in $\left(\frac{3}{2}, 2\right)$.

$\Rightarrow\ 2a \geq 2 \quad \Rightarrow\ a \geq 1$ but $a \neq 1$

$\Rightarrow\ a > 1$

Hence the final answer is $\left(\frac{1}{2}, \frac{3}{4}\right] \cup (1, \infty)$.

Problem 13. For $x \in (0, 1)$, prove that $x - \frac{x^3}{3} < \tan^{-1}x < x - \frac{x^3}{6}$. Hence or otherwise find $\lim\limits_{x\to 0}\left[\frac{\tan^{-1}x}{x}\right]$.

Solution Let $f(x) = x - \frac{x^3}{3} - \tan^{-1}x$

$f'(x) = 1 - x^2 - \frac{1}{1+x^2}$

$f'(x) = -\frac{x^4}{1+x^2}$

$f'(x) < 0$ for $x \in (0, 1)$

$\Rightarrow$ f(x) is strictly decreasing

$\Rightarrow$ f(x) < f(0)

$\Rightarrow\ x - \frac{x^3}{3} - \tan^{-1}x < 0$

$\Rightarrow\ x - \frac{x^3}{3} < \tan^{-1}x \qquad ...(1)$

Similarly, $g(x) = x - \frac{x^3}{6} - \tan^{-1}x$

$g'(x) = 1 - \frac{x^2}{2} - \frac{1}{1+x^2}$

$g'(x) = \frac{x^2(1-x^2)}{2(1+x^2)}$

$g'(x) > 0$ for $x \in (0, 1)$

$\Rightarrow$ g(x) is strictly increasing.

$\Rightarrow$ g(x) > g(0)

$x - \frac{x^3}{6} - \tan^{-1}x > 0$

$x - \frac{x^3}{6} > \tan^{-1}x \qquad ...(2)$

From (1) and (2), we get

$x - \frac{x^3}{3} < \tan^{-1}x < x - \frac{x^3}{6}$.

Also, $1 - \frac{x^2}{3} < \frac{\tan^{-1}x}{x} < 1 - \frac{x^2}{6}$

Hence by Sandwich theorem we can prove that

$\lim\limits_{x\to 0}\frac{\tan^{-1}x}{x} = 1$ but it must also be noted that as $x \to 0$, the value of $\frac{\tan^{-1}x}{x} \longrightarrow 1$ from left hand side

i.e. $\frac{\tan^{-1}x}{x} < 1$

$\Rightarrow\ \lim\limits_{x\to 0}\left[\frac{\tan^{1}x}{x}\right] = 0.$

Problem 14. If $H(x_0) = 0$ for some $x = x_0$ and $\frac{d}{dx}H(x) > 2cx\,H(x)$ for all $x \geq x_0$, where $c > 0$, then prove that H(x) cannot be zero for any $x > x_0$.

Solution Given that $\frac{d}{dx}H(x) > 2\,cx\,H(x)$

$\Rightarrow\ \frac{d}{dx}H(x) - 2\,cx\,H(x) > 0$

$\Rightarrow\ \left\{\frac{d}{dx}H(x)\right\}e^{-cx^2} - 2\,cx\,e^{-cx^2}a\,.H(x) > 0$

$\Rightarrow\ \left\{\frac{d}{dx}H(x)\right\}e^{-cx^2} + H(x)\left\{\frac{d}{dx}e^{-cx^2}\right\} > 0$

$\Rightarrow\ \left\{\frac{d}{dx}H(x).e^{-cx^2}\right\} > 0$

$\therefore\ H(x)\ e^{-cx^2}$ is an increasing function.

But, $H(x_0) = 0$ and e^{-cx^2} is always positive.

$\Rightarrow\ H(x_0) > 0$ for all $x > x_0$

$\Rightarrow$ H(x) cannot be zero for any $x > x_0$.

Problem 15. Prove that $\ell n\left(1+\frac{1}{x}\right) > \frac{1}{1-x^2}, x > 0.$

Hence, show that the function $f(x) = \left(1+\frac{1}{x}\right)^2$ strictly increases in $(0, \infty)$.

Solution Consider the function

$g(x) = \ell n\left(1+\frac{1}{x}\right) > \frac{1}{1-x}\quad \forall\, x > 0.$

$$g'(x) = \frac{-1/x^2}{1+\frac{1}{x}} + \frac{1}{(1+x)^2} = \frac{-1}{x(1+x)} + \frac{1}{(1+x)^2}$$

$$= \frac{-1}{x(1+x)} < 0 \ \forall \ x > 0$$

$\Rightarrow$ g(x) strictly decreases in $(0, \infty)$

$\Rightarrow$ $g(x) > \lim_{x\to\infty} g(x) = 0$

i.e. $\lambda n\left(1+\frac{1}{x}\right) > \frac{1}{x+1}$ ...(1)

which gives the desired result.

Now, we have

$$f(x) = \left(1+\frac{1}{x}\right)^x, x > 0 \text{ and}$$

$$f'(x) = \left(1+\frac{1}{x}\right)^x \ell n\left(1+\frac{1}{x}\right) + x\left(1+\frac{1}{x}\right)^{x-1}\left(\frac{-1}{x^2}\right)$$

$$= \left(1+\frac{1}{x}\right)^x \left[\ell n\left(1+\frac{1}{x}\right) - \frac{1}{1+x}\right] > 0 \ \forall \ x > 0$$

[using result (1)]

$\Rightarrow$ f(x) strictly increases in $(0, \infty)$.

Problem 16. Using calculus establish the inequality, $(x^b + y^b)^{1/b} < (x^a + y^a)^{1/a}$, where $x > 0$, $y > 0$ and $b > a > 0$.

Solution $(x^b + y^b)^{1/b} < (x^a + y^a)^{1/a}$

$$\Rightarrow \left(\left(\frac{x}{y}\right)^b + 1\right)^{1/b} < \left(\left(\frac{x}{y}\right)^a + 1\right)^{1/a} \quad [\because y > 0]$$

To prove that $(t^b + 1)^{a/b} < t^a + 1$ $\left[\frac{x}{y} = t > 0\right]$

Let $f(t) = (t^b + 1)^{a/b} - t^a - 1$

$$\Rightarrow f'(t) = \frac{a}{b}\left(t^b + 1\right)^{\frac{a}{b}-1} . b t^{b-1} - a t^{a-1}$$

$$\Rightarrow f'(t) = a t^{a-1}\left[t^{b-a}\left(t^b + 1\right)^{\frac{a}{b}-1} - 1\right]$$

$$= a t^{a-1}\left[\left(1 + \frac{1}{t^b}\right)^{\frac{a}{b}-1} - 1\right]$$

Now since $1 + \frac{1}{t^b} > 1$ and $\frac{a}{b} - 1 < 0$

therefore $\left(1 + \frac{1}{t^b}\right)^{\frac{a}{b}-1} < 1$.

Hence $f'(t) < 0$ i.e. $f(t)$ is decreasing function

So $f(t) < f(0)$ but $f(0) = 0$

$\Rightarrow$ $(t^b + 1)^{a/b} < t^a + 1$.

Hence proved

Problem 17. Prove that $f(x) = \left(1+\frac{1}{x}\right)^x$ is strictly increasing in its domain. Hence, draw the graph of f(x) and find its range.

Solution $f(x) = \left(1+\frac{1}{x}\right)^x$

Domain of f(x) : $1 + \frac{1}{x} > 0$

$\Rightarrow \frac{x+1}{x} > 0 \quad \Rightarrow (-\infty, -1) \cup (0, \infty)$

Consider $f'(x) = \left(1+\frac{1}{x}\right)^x \left[\ell n\left(1+\frac{1}{x}\right) + \frac{x}{1+\frac{1}{x}}\frac{-1}{x^2}\right]$

$$\Rightarrow f'(x) = \left(1+\frac{1}{x}\right)^x \left[\ell n\left(1+\frac{1}{x}\right) - \frac{1}{x+1}\right]$$

Now $\left(1+\frac{1}{x}\right)^x$ is always positive, hence the sign of

$f'(x)$ depends on sign of $\ell n\left(1+\frac{1}{x}\right) - \frac{1}{1+x}$

i.e. we have to compare $\ell n\left(1+\frac{1}{x}\right)$ and $\frac{1}{1+x}$

So let us assume $g(x) = \ell n\left(1+\frac{1}{x}\right) - \frac{1}{1+x}$

$$g'(x) = \frac{1}{1+\frac{1}{x}}\frac{-1}{x^2} + \frac{1}{(x+1)^2}$$

$$\Rightarrow g'(x) = \frac{-1}{x(x+1)^2}$$

For $x \in (0, \infty)$, $g'(x) < 0$ ...(1)

$\Rightarrow$ g(x) is strictly decreasing for $x \in (0, \infty)$

$g(x) > \lim_{x\to\infty} g(x)$

$g(x) > 0$.

and since $g(x) > 0 \Rightarrow f'(x) > 0$

For $x \in (-\infty, -1)$, $g'(x) > 0$ ...(3)

$\Rightarrow$ g(x) is strictly increasing for $x \in (-\infty, -1)$

$\Rightarrow$ $g(x) > \lim_{x\to-\infty} g(x)$

$\Rightarrow$ $g(x) > 0 \Rightarrow f'(x) > 0$

Hence from (1) and (2) we get $f'(x) > 0$ for all $x \in (-\infty, -1) \cup (0, \infty)$

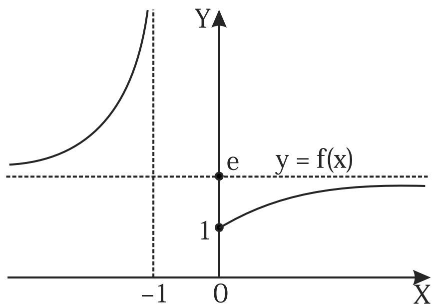
$\Rightarrow$ f(x) is strictly increasing in its domain

For drawing the graph of f(x), its important to find the value of f(x) at endpoints $\pm\infty$, 0, –1.

$$\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x = e$$

$$\lim_{x\to 0^+}\left(1+\frac{1}{x}\right)^x = 1 \quad \text{and} \quad \lim_{x\to -1}\left(1+\frac{1}{x}\right)^x = \infty$$

So the graph of f(x) is



From the graph, the range is $y \in (1, \infty) - \{e\}$.

Problem 18. Let R^+ be the set of positive real numbers. Find all functions $f : R^+ \to R^+$ such that for all $x, y \in R^+$, $f(x)\, f(yf(x)) = f(x + y)$.

Solution First, if we assume that $f(x) > 1$ for some $x \in R^+$, setting $y = \dfrac{x}{f(x) - 1}$ gives the contradiction $f(x) = 1$. Hence $f(x) \le 1$ for each $x \in R^+$, which implies that f is a decreasing function.

If $f(x) = 1$ for some $x \in R^+$, then $f(x + y) = f(y)$ for each $y \in R^+$, and by the monotonicity of f it follows that $f \equiv 1$.

Let now $f(x) < 1$ for each $x \in R^+$. Then f is strictly decreasing function, in particular injective. By the equalities

$$f(x)\, f(yf(x)) = f(x + y) = f(yf(x) + x + y(1 - f)))$$
$$= f(yf(x)))\, f((x + y(1 - f(x)))\, f(yf(x))))$$

We obtain that $x = (x + y(1 - f(x)))\, f(yf(x))$. Setting $x = 1$, $z = xf(1)$ and $a = \dfrac{1 - f(1)}{f(1)}$, we get $f(z) = \dfrac{1}{1 + az}$.

Combining the two cases, we conclude that $f(x) = \dfrac{1}{1 + ax}$ for each $x \in R^+$, where $a \ge 0$. Conversely, a direct verification shows that the function of this form satisfy the initial equality.

Alternative : As in the first solution we get that f is a decreasing function, in particular differentiable almost everywhere. Write the initial equality in the form

$$\frac{f(x + y) - f(x)}{y} = f^2(x)\frac{f(yf(x)) - 1}{yf(x)}.$$

It follows that if f is differentiable at the point $x \in R^+$, then there exists the limit $\lim_{z\to 0^+}\dfrac{f(z) - 1}{z} = -a$. Therefore $f'(x) = -af^2(x)$ for each $x \in R^+$, i.e. $\left(\dfrac{1}{f(x)}\right)' = a$, which means that $f(x) = \dfrac{1}{ax + b}$. Substituting in the initial relation, we find that $b = 1$ and $a \ge 0$.

Problem 19. Does there exist a continuously differentiable function $f : R \to R$ such that for every $x \in R$ we have $f(x) > 0$ and $f'(x) = f(f(x))$?

Solution Assume that there exists such a function. Since $f'(x) = f(f(x)) > 0$, the function is strictly increasing. By the monotonicity, $f(x) > 0$ implies $f(f(x)) > f(0)$ for all x. Thus f(0) is a lower bound for $f'(x)$, and for all $x < 0$ we have $f(x) < f(0) + x$. $f(0) = (1 + x)\, f(0)$. Hence, if $x \le -1$ then $f(x) \le 0$, contradicting the property $f(x) > 0$. So, such a function does not exist.

Problem 20. Let $f : R \to R$ be a real function. Prove or disprove each of the following statements.

(a) If f is continuous and range (f) = R then f is monotonic.

(b) If f is monotonic and range (f) = R then f is continuous.

(c) If f is monotonic and f is continuous then range (f) = R.

Solution

(a) False. Consider function $f(x) = x^3 - x$. It is continuous, range (f) = R but, for example, $f(0) = 0$, $f\left(\dfrac{1}{2}\right) = -\dfrac{3}{8}$ and $f(1) = 0$, therefore $f(0) > f\left(\dfrac{1}{2}\right)$, $f\left(\dfrac{1}{2}\right) < f(1)$ and f is not monotonic.

(b) True, Assume first that f is non-decreasing. For an arbitrary number a, the limits $\lim_{x\to a^-} f$ and $\lim_{x\to a^+} f$ exist and $\lim_{x\to a^-} f \le \lim_{x\to a^+} f$. If the two limits are equal, the function is continuous at a. Otherwise, if $\lim_{x\to a^-} f = b < \lim_{x\to a^+} f = c$, we have $f(x) \le b$ for all $x < a$ and $f(x) \ge c$ for all $x > a$;
therefore range $(f) \subset (-\infty, b) \cup (c, \infty) \cup \{f(a)\}$ cannot be the complete R.
For non-increasing f the same can be applied writing reverse relations or $g(x) = -f(x)$.

(c) False. The function $g(x) = \tan^{-1}x$ is monotonic and continuous, but range $(g) = (-\pi/2, \pi/2) \neq R$.

Problem 21. Let $f: (a, b) \to R$, $\lim_{x \to a^+} f(x) = \infty$, $\lim_{x \to a^-} f(x) = -\infty$ and $f'(x) + f^2(x) \geq -1$ for $x \in (a, b)$. Prove that $b - a \geq \pi$ and give an example where $b - a = \pi$.

Solution From the inequality we get

$$\frac{d}{dx}(\tan^{-1}x\, f(x) + x) = \frac{f'(x)}{1 + f^2(x)} + 1 \geq 0$$

for $x \in (a, b)$. Thus $\tan^{-1}f(x) + x$ is non-decreasing in the interval and using the limits we get

$$\frac{\pi}{2} + a \leq -\frac{\pi}{2} + b.$$

Hence $b - a \geq \pi$. One has equality for

$f(x) = \cot x$, $a = 0$, $b = \pi$.

Problem 22. Let $f \in c'[a, b]$, $f(a) = 0$ and suppose that $\lambda \in R$, $\lambda > 0$, is such that $|f'(x)| \leq \lambda|f(x)|$ for all $x \in [a, b]$. Is it true that $f(x) = 0$ for all $x \in [a, b]$?

Solution Assume that there is $y \in (a, b]$ such that $f(y) \neq 0$. Without loss of generality we have $f(y) > 0$.

In view of the continuity of f there exists $c \in [a, y)$ such that $f(c) = 0$ and $f(x) > 0$ for $x \in (c, y]$.

For $x \in (c, y]$ we have $|f'(x)| \leq \lambda f(x)$. This implies that the function $g(x) = \ln f(x) - \lambda x$ is not increasing in $(c, y]$ because of $g'(x) = \dfrac{f'(x)}{f(x)} - \lambda \leq 0$.

Thus, $\ln f(x) - \lambda x \geq \ln f(y) - \lambda y$ and $f(x) \geq e^{\lambda x - \lambda y} f(y)$ for $x \in (c, y]$.

Thus, $0 = f(c) = f(c + 0) \geq e^{\lambda c - \lambda y} f(y) > 0$

a contradiction. Hence one has $f(x) = 0$ for $x \in [a, b]$.

Problem 23. Suppose that $f(x)$ is a real-valued function defined for real values of x. Suppose that both $f(x) - 3x$ and $f(x) - x^3$ are increasing functions. Must $f(x) - x - x^2$ also be increasing on all of the real numbers or on at least the positive reals ?

Solution Let $u \geq v$. Suppose that $u + v \leq 2$. Then, since $f(x) - 3x$ is increasing,

$f(u) - 3u \geq f(v) - 3v,$

$\Rightarrow f(u) - f(v) \geq 3(u - v) \geq (u + v + 1)(u - v)$

$= u^2 - v^2 + u - v$

$\Rightarrow f(u) - u - u^2 \geq f(v) - v - v^2$

Suppose that $u + v \geq 2$. Then, since $f(x) - x^3$ is increasing,

$f(u) - u^3 \geq f(v) - v^3 \Rightarrow f(u) - f(v) \geq u^3 - v^3$

$= (u - v)(u^2 + uv + v^2).$

Now $2[(u^2 + uv + v^2) - (u + v + 1)]$

$= (u + v)^2 + (u - 1)^2 + (v - 1)^2 - 4 \geq 0,$

so that $u^2 + uv + v^2 \geq u + v + 1$ and

$f(u) - f(v) \geq (u - v)(u + v + 1)$

$= u^2 - v^2 + u - v \Rightarrow f(u) - u - u^2 \geq f(v) - v - v^2.$

Hence $f(u) - u - u^2 \geq f(v) - v - v^2$ whenever $u \geq v$, so that $f(x) - x - x^2$ is increasing.

Things to Remember

1. A function f(x) is said to be strictly increasing about the point $x = a$ if $f(a - h) < f(a) < f(a + h)$, where h is a small positive arbitrary number.
2. A function f(x) is said to be strictly decreasing about the point $x = a$ if $f(a - h) > f(a) > f(a + h)$, where h is a small positive arbitrary number.
3. A function f(x) is said to be non-decreasing about the point $x = a$ if $f(a - h) \leq f(a) \leq f(a + h)$, where h is a small positive arbitrary number.
4. A function f(x) is said to be non-decreasing about the point $x = a$ if $f(a - h) \geq f(a) \geq f(a + h)$, where h is a small positive arbitrary number.
5. Let a function f be differentiable at $x = a$.
   (i) If $f'(a) > 0$ then f(x) is strictly increasing at $x = a$.
   (ii) If $f'(a) < 0$ then f(x) is strictly decreasing at $x = a$.
   (iii) If $f'(a) = 0$ then we need to examine the signs of $f'(a - h)$ and $f'(a + h)$.
   (a) If $f'(a - h) > 0$ and $f'(a + h) > 0$ then f(x) is strictly increasing at $x = a$.
   (b) If $f'(a - h) < 0$ and $f'(a + h) < 0$ then f(x) is strictly decreasing at $x = a$.
   (c) If $f'(a - h)$ and $f'(a + h)$ have opposite signs then f(x) is neither increasing nor decreasing (non-monotonous) at $x = a$.
6. Assume that the function f is differentiable at $x = a$.
   (a) If $x = a$ is the left endpoint, we check as follows:
   (i) If $f'(a^+) > 0$, then f(x) is strictly increasing at $x = a$.

(ii) If $f'(a^+) < 0$, then f(x) is strictly decreasing at $x = a$.
(iii) If $f'(a^+) = 0$, but $f'(a + h) > 0$, then f(x) is strictly increasing at $x = a$.
(iv) If $f'(a^+) = 0$, but $f'(a + h) < 0$, then f(x) is strictly decreasing at $x = a$.

(b) If $x = a$ is the right endpoint, we check as follows:
(i) If $f'(a^-) > 0$, then f(x) is strictly increasing at $x = a$.
(ii) If $f'(a^-) < 0$, then f(x) is strictly decreasing at $x = a$.
(iii) If $f'(a^-) = 0$, but $f'(a - h) > 0$, then f(x) is strictly increasing at $x = a$.
(iv) If $f'(a^-) = 0$, but $f'(a - h) < 0$, then f(x) is strictly decreasing at $x = a$.

7. Necessary Conditions for Monotonicity
(i) If a differentiable function f(x) increases in an interval its derivative $f'(x)$ is non-negative : $f'(x) \geq 0$.
(ii) If a differentiable function f(x) decreases in an interval its derivative $f'(x)$ is non-positive: $f'(x) \leq 0$.
(iii) If a differentiable function f(x) does not vary in an interval (i.e. is equal to a constant) its derivative is identically equal to zero: $f'(x) = 0$.

8. Sufficient Conditions for Monotonicity
Let f(x) be a differentiable function on the interval (a, b). Then :
(i) If the derivative $f'(x)$ is everywhere positive (i.e. $f'(x) > 0$) in the interval (a, b), then the function f(x) is strictly increasing in the interval (a, b).
(ii) If the derivative $f'(x)$ is everywhere negative (i.e. $f'(x) < 0$) in the interval (a, b), then the function f(x) is strictly decreasing in the interval (a, b).
(iii) If the derivative $f'(x)$ is everywhere equal to zero in the interval (a, b), then the function f(x) does not vary in the interval (a, b) (i.e. it is constant).

9. Monotonicity at points where $f'(x)$ does not exist
Consider a continuous function f(x) whose derivative $f'(x)$ does not exist at $x = c$ but exists in the neighbourhood of c.
(i) If $f'(c^-) > 0$ and $f'(c^+) > 0$, then f(x) is strictly increasing at $x = c$.
(ii) If $f'(c - h) > 0$, $f'(c^-) \geqq 0$, $f'(c^+) \geqq 0$, $f'(c + h) > 0$, then f(x) is strictly increasing at $x = c$.
(iii) If $f'(c^-) < 0$ and $f'(c^+) < 0$, then f(x) is strictly decreasing at $x = c$.
(iv) If $f'(c - h) > 0$, $f'(c^-) \leqq 0$, $f'(c^+) \leqq 0$, $f'(c + h) > 0$, then f(x) is strictly increasing at $x = c$.

10. A critical point of a function f is a number c in the domain of f such the either $f'(c) = 0$ or $f'(c)$ does not exist.

11. Steps for finding intervals of monotonicity
(i) Compute the derivative $f'(x)$ of a given function f(x), and then find the points at which $f'(x)$ equals zero or does not exist at all. These points are the critical points for the function f(x).
(ii) Using the critical points, separate the domain of definition of the function f(x) into several intervals on each of which the derivative $f'(x)$ retains its sign. These intervals will be the intervals of monotonicity.
(iii) Investigate the sign of $f'(x)$ on each of the found intervals. If on a certain interval $f'(x) > 0$, then the function f(x) increases on this interval, and if $f'(x) < 0$, then f(x) decreases on this interval.

12. Application of monotonicity in isolation of roots
Suppose that
(i) f is continuous on [a, b] and differentiable on (a, b).
(ii) f(a) and f(b) have opposite signs,
(iii) $f'(x) > 0$ on (a, b) or $f'(x) < 0$ on (a, b).
Then f has exactly one root between a and b.

13. (i) If f(x) is a strictly increasing function then its negative $g(x) = -f(x)$ is a strictly decreasing function and vice-versa.
(ii) The reciprocal of a nonzero strictly increasing function is a strictly decreasing function and vice-versa.
(iii) If f and g are strictly increasing functions then $h(x) = f(x) + g(x)$ is also a strictly increasing function.
(iv) If f and g are positive and both are strictly increasing then $h(x) = f(x) \times g(x)$ is also strictly increasing.
(v) If f is strictly increasing in [a, b] and g is strictly increasing in [f(a), f(b)], then gof is strictly increasing in [a, b].
(vi) If f is strictly decreasing in [a, b] and g is strictly decreasing in [f(b), f(a)], then gof is strictly increasing in [a, b].
(vii) If f is strictly increasing in [a, b] and g is strictly

decreasing in [f(a), f(b)], then gof is strictly decreasing in [a, b].

(viii) If f is strictly decreasing in [a, b] and g is strictly increasing in [f(b), f(a)], then gof is strictly decreasing in [a, b].

14. If the second derivative $f''(x)$ is everywhere positive within an interval the arc of the curve $y = f(x)$ corresponding to that interval is concave up. If the second derivative $f''(x)$ is everywhere negative in an interval, the corresponding arc of the curve $y = f(x)$ is concave down.

15. A hyper-critical point of a function f is a number c in the domain of f such the either $f''(c) = 0$ or $f''(c)$ does not exist.

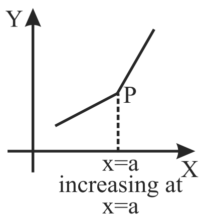
16. Suppose the graph of a function f has a tangent line (possibly vertical) at the point P(c, f(c)) and that the graph is concave up on one side of P and concave down on the other side. Then P is called an inflection point of the graph.

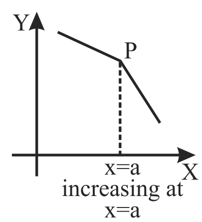
17. If $x = c$ is a hyper-critical point and the inequalities $f''(c - h) < 0$, $f''(c + h) > 0$ (or inequalities $f''(c - h) > 0$, $f''(c + h) < 0$) hold for an arbitrary sufficiently small $h > 0$, then the point of the curve $y = f(x)$ with the abscissa $x = c$ is a point of inflection. If $f''(c - h)$ and $f''(c + h)$ are of the same sign, then the point $x = c$ is not a point of inflection.

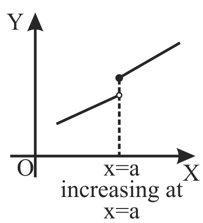
Objective Exercises

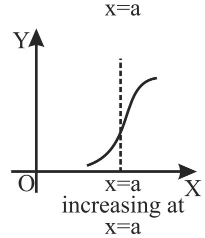
SINGLE CORRECT ANSWER TYPE

1. Which of the following conclusions does not hold true?

(A)


(B)


(C)


(D)


2. If $f(x) = \sin^2 x - 3\cos^2 x + 2ax - 4$ is increasing for all $x \geq 0$, then a is an element of

(A) $[-2, 0)$ (B) $(-\infty, -2]$
(C) $[2, \infty)$ (D) $(-\infty, 2]$

3. Let $f(x) = \sin^{-1}\left(\frac{2\varphi(x)}{1+\varphi^2(x)}\right)$, where $\phi(x)$ is a decreasing function of x, then

(A) f(x) increasing when $|\phi(x)| < 1$
(B) f(x) is decreasing when $|\phi(x)| < 1$
(C) f(x) is decreasing always
(D) f(x) is increasing always

4. Let $f(x) = \sin^2 x - (2a+1)\sin x + (a-3)$. If $f(x) \leq 0$ for all $x \in \left[0, \frac{\pi}{2}\right]$, then range of values of a is

(A) $[-3, 0]$ (B) $[3, \infty)$
(C) $[-3, 3]$ (D) $(-\infty, 3]$

5. Let f be a function such that f(x) and $f'(x)$ have opposite signs for all $x \in R$. Then

(A) f(x) is an increasing function
(B) f(x) is a decreasing function
(C) | f(x)| is an decreasing function
(D) |f(x)| is an increasing function

6. If $f(x) = \sin\left(\sin^{-1}\left(\log_{\frac{1}{2}} x\right)\right)$, then

(A) Domain of f(x) is $\left(0, \frac{1}{2}\right]$
(B) Range of f(x) is $[-1,1]$
(C) f(x) is decreasing in its domain
(D) f(x) is increasing in its domain

7. The function $f : R \to R$ is such that $f(x) = a_1x + a_3x^3 + a_5x^5 + \ldots\ldots + a_{2n+1}x^{2n+1} - \cot^{-1}x$ where $0 < a_1 < a_2 < \ldots\ldots a_{2n+1}$ then the function f(x) is

(A) one-one into (B) many one into
(C) one-one onto (D) many one onto

8. If f(x) is a differentiable real valued function satisfying $f''(x) - 3f'(x) > 3 \ \forall \ x \geq 0$ and $f'(0) = -1$, then $f(x) + x \ \forall \ x > 0$ is

(A) decreasing function of x
(B) increasing function of x
(C) constant function
(D) none of these

9. Let $f(x) = ax + \sin 2x + b$, then $f(x) = 0$ has
(A) exactly one positive real root, if $a > 2, b < 0$
(B) exactly one positive real root, if $a > 2, b > 0$
(C) infinite number of positive real root, if $a < -2$
(D) None of these

10. $f : R \to R$ be a differentiable function $\forall\ x \in R$. If a tangent drawn to the curve at any point $x \in (a, b)$ always lies below the curve, then
(A) $f'(x) < 0$ and $f''(x) > 0\ \forall\ x \in (a, b)$
(B) $f'(x) > 0$ and $f''(x) < 0\ \forall\ x \in (a, b)$
(C) $f'(x)$ can have any value and $f''(x) > 0\ \forall\ x \in (a, b)$
(D) none of these

11. The set of value of a, for which the function $f(x) = (4a - 3)(x + 5) + 2(a - 7) \cot (x/2) \sin^2(x/2)$ does not possess any critical point is given by
(A) $(-\infty, -4/3)$ (B) $(-\infty, -1)$
(C) $(-4/3, 2)$ (D) $(-\infty, -4/3) \cup (2, \infty)$

12. If $\tan (\pi \cos \theta) = \cot (\pi \sin \theta)$ $0 < \theta < \frac{\pi}{2}$ and $f(x) = (\cos \theta + \sin \theta)^x$, then f is
(A) increasing for all $x \in R$
(B) decreasing for all $x \in R$
(C) increasing in $(0, \infty)$
(D) decreasing in $(0, \infty)$

13. If $f(x) = x^n \sin 1/x + x^m \cos \frac{1}{2x}$, then
(A) atleast one root of $f'(x) = 0$ will lie in interval $[1/\pi, 2/\pi]$
(B) atleast one root of $f'(x) = 0$ will lie in interval $[1/\pi, 1/3\pi]$
(C) atleast one root of $f'(x) = 0$ will lie in interval $[1/\pi, 1/2\pi]$
(D) None of these

14. Which one is correct ?
(A) $(1999)^{2000} > (2000)^{1999}$
(B) $(1998)^{1999} > (1999)^{1998}$
(C) $(100)^{101} < (101)^{100}$
(D) $26^{25} > 25^{26}$

15. Let $f''(x) > 0\ \forall\ x \in R$ and $g(x) = f(2 - x) + f(4 + x)$. Then g(x) is increasing in
(A) $(-\infty, -1)$ (B) $(-\infty, 0)$
(C) $(-1, \infty)$ (D) None of these

16. Let the sign scheme of $f'(x)$ of a differentiable function f be
(A) If $g(x) = f(x) + 5$ then $g'(0) > 0$
(B) If $g(x) = -f(x)$ then $g'(-6) > 0$
(C) If $g(x) = f(x - 10)$ then $g'(0) > 0$
(D) none of these

17. Let f be differentiable on [a, b] when $f(A) = f(B) = 0$ and $f'(C) = 0$, $a < c < b$. Then the incorrect statement is
(A) If $g(x) = k\, f(x)$ then $g'(C) = 0$
(B) If $g(x) = f(x - k)$ then $g'(c + k) = 0$
(C) If $g(x) = f(kx)$ then $g'(c/k) = 0$
(D) none of these

18. Let f (x) and g (x) are two function which are defined and differentiable for all $x \geq x_0$. If $f(x_0) = g(x_0)$ and $f'(x) > g'(x)$ for all $x > x_0$ then
(A) $f(x) < g(x)$ for some $x > x_0$
(B) $f(x) = g(x)$ for some $x > x_0$
(C) $f(x) > g(x)$ only for some $x > x_0$
(D) $f(x) > g(x)$ for all $x > x_0$

19. The number of zeros of the cubic $f(x) = x^3 + 2x + k\ \forall\ k \in R$, is
(A) 0 (B) 1
(C) 2 (D) 3

20. Let f, g and h are differentiable function such that $g(x) = f(x) - x$ and $h(x) = f(x) - x^3$ are both strictly increasing functions, then the function $F(x) = f(x) - \frac{\sqrt{3}\, x^2}{2}$ is
(A) strictly increasing $\forall\ x \in R$
(B) strictly decreasing $\forall\ x \in R$
(C) strictly decreasing on $\left(-\infty, \frac{1}{\sqrt{3}}\right)$ and strictly increasing on $\left(\frac{1}{\sqrt{3}}, \infty\right)$
(D) strictly increasing on $\left(-\infty, \frac{1}{\sqrt{3}}\right)$ and strictly

decreasing on $\left(\frac{1}{\sqrt{3}}, \infty\right)$

21. Let $f : R \to R$ be a real function. Then
(A) If f is continuous and range (f) = R then is
(B) If f is monotonic and range (f) = R then f is continuous
(C) If f is monotonic and continuous then range f(x) = R
(D) None of these

22. The number of inflection points on the curve represented parametrically by the equations $x = t^2$, $y = 3t + t^3$ is
(A) 0 (B) 1
(C) 2 (D) 3

23. If $f : R \to R$ is the function defined by
$f(x) = \frac{e^{x^2} - e^{-x^2}}{e^{x^2} + e^{-x^2}}$, then
(A) f(x) is an increasing function
(B) f(x) is a decreasing function
(C) f(x) is onto
(D) None of these

24. Let domain and range of f(x) and g(x) are $[0, \infty)$. If f(x) be an increasing and g(x) be decreasing function. Also h(x) = f(g(x)), h(0) = 0 and $p(x) = h(x^3 - 2x^2 + 2x)$. h(4) then for all x belonging to (0, 2) (here f and g both are continuous & differentiable functions)
(A) $p(x) \in (0, -h(4))$
(B) $p(x) \in (-h(4), 0)$
(C) $p(x) \in (-h(4), h(4))$
(D) $p(x) \in (h(4), -h(4))$

25. Let f(x) be an increasing function and
$f(x) \le -1 \ \forall \ x \in R$ then $g(x) = \frac{f(x)}{1+x^2}$ is
(A) increasing for all x
(B) decreasing for $x > 0$
(C) increasing for $x > 0$
(D) None of these

26. If $f'(x^2 - 4x + 3) > 0 \ \forall \ x \in (2, 3)$, then f(sin x) is increasing on
(A) $(n\pi, n\pi/2)$, $n \in I$
(B) $((2n+1)\pi, (4n+3)\pi/2)$, $n \in I$
(C) $((4n-1)\pi/2, 2n\pi)$, $n \in I$
(D) None of these

27. A function f(x) is given by $x^2 f'(x) + 2x f(x) - x + 1 = 0$ $(x \ne 0)$. If $f(1) = 0$ then f(x) is
(A) increasing in $(-\infty, 0)$, $(1, \infty)$ and decreasing in $(0, 1)$
(B) increasing in (0, 1) and decreasing in $(-\infty, 0)$, $(1, \infty)$
(C) increasing in $(-\infty, 0)$ and decreasing in $(0, \infty)$
(D) increasing in $(0, \infty)$ and decreasing in $(-\infty, 0)$

28. Let $f(x) = x\sqrt{4ax - x^2}$, $(a > 0)$. Then f(x) is
(A) increasing in (0, 3a), decreasing in $(-\infty, 0)$ and $(3a, \infty)$
(B) increasing in (a, 4a), decreasing in $(4a, \infty)$
(C) increasing in (0, 4a), decreasing in $(-\infty, 0)$
(D) none of these

29. If $f'(x) = |x| - \{x\}$, where {.} denotes the fractional part of x, then f(x) is decreasing in
(A) $\left(-\frac{1}{2}, 0\right)$ (B) $\left(-\frac{1}{2}, 2\right)$
(C) $\left(-\frac{1}{2}, 2\right)$ (D) $\left(\frac{1}{2}, \infty\right)$

30. Which of the following statements is true for the function
$$f(x) = \begin{cases} \sqrt{x}, & x \ge 1 \\ x^3, & 0 \le x \le 1 \\ \frac{x^3}{3} - 4x, & x < 0 \end{cases}$$
(A) f is strictly increasing $\forall \ x \in R$.
(B) $f'(x)$ fails to exist at 3 distinct values of x
(C) $f'(x)$ changes its sign twice as x varies from $-\infty$ to ∞.
(D) f attains its extreme values at x_1 and x_2, where $x_1 x_2 > 0$

31. If $f''(x) > 0$, $\forall \ x \in R$, $f'(3) = 0$ and $g(x) = f(\tan^2 x - 2\tan x + 4)$, $0 < x < \frac{\pi}{2}$, then g(x) is increasing in
(A) $\left(0, \frac{\pi}{4}\right)$ (B) $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$
(C) $\left(0, \frac{\pi}{3}\right)$ (D) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

32. Let f be a function such that $f'(x) = \log_{1/3}(\log_3(\sin x + a))$. If f is decreasing for all real values of x, then

(A) $a \in (1,4)$ (B) $a \in (4, \infty)$
(C) $a \in (2,3)$ (D) $a \in (2,\infty)$

33. If $f(x) = x + \sin x$, $g(x) = e^{-x}$, $u = \sqrt{c+1} - \sqrt{c}$, $v = \sqrt{c} - \sqrt{c-1}$, $(c > 1)$ then

(A) $fog(u) \ge fog(v)$ (B) $gof(u) \le gof(v)$
(C) $gof(u) > gof(v)$ (D) $fog(u) < fog(v)$

34. The length of the largest continuous interval in which the function $f(x) = 4x - \tan 2x$ is monotonic is

(A) $\pi/2$ (B) $\pi/4$
(C) $\pi/8$ (D) $\pi/16$

35. The number of solutions of the equation $x^3 + 2x^2 + 5x + 2\cos x = 0$ in $[0, 2\pi]$ is

(A) one (B) two
(C) three (D) zero

36. If $f(x) = x^3 + 4x^2 + \lambda x + 1$ is a strictly decreasing function of x in the largest possible interval $[-2, 2/3]$ then

(A) $\lambda = 4$ (B) $\lambda = 2$
(C) $\lambda = -1$ (D) λ has no real value

37. Let $g(x) = \frac{1}{4}f(2x^2 - 1) + \frac{1}{2}f(1 - x^2)$ where $f'(x)$ is an increasing function, then g(x) is increasing in the interval

(A) $(-1, 1)$
(B) $\left(-\sqrt{\frac{2}{3}}, 0\right) \cup \left(\sqrt{\frac{2}{3}}, \infty\right)$
(C) $\left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$
(D) None of these

38. If $f(x) = (ab - b^2 - 2)x + \int_0^x (\cos^4\theta + \sin^4\theta)\,d\theta$ is a decreasing function of x for all $x \in R$ and $b \in R$, then

(A) $a \in (0, \sqrt{6})$ (B) $a \in (-\sqrt{6}, \sqrt{6})$
(C) $a \in (-\sqrt{6}, \sqrt{6})$ (D) None of these

39. Let f(x) be an increasing function and

$$g(\theta) = \int_0^{\sin^2\theta} f(x)dx + \int_0^{\cos^2\theta} f(x)dx,$$

$x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

then $g(\theta)$ is increasing in the interval

(A) $\left(-\frac{\pi}{2}, 0\right)$ (B) $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$
(C) $\left(0, \frac{\pi}{4}\right)$ (D) $\left(-\frac{\pi}{4}, 0\right)$

40. The number of solutions of the equation $x^3 + 2x^2 + 5x + 2\cos x = 0$ in $[0, 2\pi]$ is

(A) 3 (B) 2
(C) 1 (D) 0

41. If $f(x) = ax^3 + bx^2 + cx + d$ where a, b, c, d are real numbers and $3b^2 < c^2$, is an increasing cubic function and $g(x) = af'(x) + bf''(x) + c^2$, then

(A) $\int_a^x g(t)\,dt$ is a decreasing function
(B) $\int_a^x g(t)\,dt$ is an increasing function
(C) $\int_a^x g(t)\,dt$ is neither increasing nor decreasing function
(D) None of the above

42. If the function $y = \sin(f(x))$ is monotonic for all values of x (where f(x) is continuous), then the maximum value of the difference between the maximum and the minimum values of f(x), is

(A) π (B) 2π
(C) $\pi/2$ (D) None

43. The interval in which $f(x) = 3\cos^4 x + 10\cos^3 x + 6\cos^2 x - 3$, $x \in (0, \pi)$, decreases or increases are

(A) Decreases on $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ and increases on $\left(0, \frac{\pi}{2}\right), \left(\frac{2\pi}{3}, \pi\right)$
(B) Decreases on $\left(\frac{\pi}{2}, \pi\right)$ and increases on $\left(0, \frac{\pi}{2}\right)$
(C) Decreases on $\left(0, \frac{\pi}{2}\right), \left(\frac{2\pi}{3}, \pi\right)$ and increases on $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$
(D) Decreases on $\left(0, \frac{\pi}{2}\right)$ and increases on $\left(\frac{\pi}{2}, \pi\right)$

44. Let f(x) be a differentiable function such that, $f'(x) = \frac{1}{\log_3(\log_{1/4}(\cos x + a))}$. If f(x) is increasing for all values of x then

(A) $a \in (5, \infty)$ (B) $a \in \left(1, \frac{5}{4}\right)$

(C) $a \in \left(\frac{5}{4}, 5\right)$ (D) None of these

45. The intervals of monotonicity of the function of the function $f(x) = x^2 - \ln|x|$, when $(x \neq 0)$ is/are

(A) Increasing for all $x > 0$ and decreasing for all $x < 0$

(B) Increasing when $x \in \left(-\infty, -\frac{1}{\sqrt{2}}\right), \left(0, \frac{1}{\sqrt{2}}\right)$ and decreasing when $x \in \left(-\frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{2}}, \infty\right)$

(C) Increasing when $x \in \left(-\frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{2}}, \infty\right)$ and decreasing when $x \in \left(-\infty, -\frac{1}{\sqrt{2}}\right), \left(0, \frac{1}{\sqrt{2}}\right)$

(D) Increasing when $x \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) - \{0\}$ and decreasing when $x \in \left(-\infty, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \infty\right)$

46. If a function f(x) is such that $f(2) = 3$, $f'(2) = 4$, then $\lim_{x \to 2}[f(x)]$, (where [.] is G.I.F.), is

(A) 2 (B) 3

(C) 4 (D) doesn't exist

47. Let $f : R \to R$ be a differentiable function such that $f(f(x)) = a(x^5 + x)$, $(a \neq 0)$, then

(A) f(x) is strictly increasing

(B) f(x) is strictly decreasing

(C) f(x) is either strictly increasing or decreasing

(D) f(x) is non-monotonic.

48. Let $f^1(x)$ denotes g(x), $f^2(x)$ denotes g(g(x)), $f^3(x)$ denotes g(g(g(x))) and so on. If g(x) is an increasing function and lies between $(0, \infty)$, $\forall x \in R$, then $\lim_{n \to \infty}\left(\frac{1}{f^n(x)}\right)$ is equal to

(A) 0 (B) 1

(C) does not exist (D) none of these

49. f(x) is an increasing, concave up function for all $x \in [a, b]$, then for any $\lambda > 0$, (f is invertible)

(A) $\left(\frac{b + a\lambda}{1 + \lambda}\right) > f^{-1}\left(\frac{f(b) + \lambda f(a)}{1 + \lambda}\right)$

(B) $\left(\frac{b + a\lambda}{1 + \lambda}\right) \leq f^{-1}\left(\frac{f(b) + \lambda f(a)}{1 + \lambda}\right)$

(C) $\left(\frac{b + a\lambda}{1 + \lambda}\right) < f^{-1}\left(\frac{f(b) + \lambda f(a)}{1 + \lambda}\right)$

(D) $\left(\frac{b + a\lambda}{1 + \lambda}\right) \leq f^{-1}\left(\frac{f(b) + \lambda f(a)}{1 + \lambda}\right)$

50. If $g(x) = 2f(2x^3 - 3x^2) + f(6x^2 - 4x^3 - 3)$, $\forall x \in R$, and $f''(x) > 0$, $\forall x \in R$ then g(x) is increasing in the interval

(A) $\left(-\infty, -\frac{1}{2}\right), (0,1)$ (B) $\left(-\frac{1}{2}, 0\right), (1, \infty)$

(C) $(0, \infty)$ (D) none of these

MULTIPLE CORRECT ANSWER TYPE FOR JEE ADVANCED

51. Which of the following is/are true

(A) $(\ell n 2.1)^{\ell n 2.2} > (\ell n 2.2)^{\ell n 2.1}$

(B) $(\ell n 4)^{\ell n 5} < (\ell n 5)^{\ell n 4}$

(C) $(\ell n 30)^{\ell n 31} > (\ell n 31)^{\ell n 30}$

(D) $(\ell n 28)^{\ell n 30} < (\ell n 30)^{\ell n 28}$

52. If x = c is a critical point of y = f(x) and y = g(x) then x = c is also a critical point of

(A) f . g (B) f . g,

(C) f + g (D) none

53. Let $g'(x) > 0$ and $f'(x) < 0$, $\forall x \in R$, then

(A) $f(x + 1) > g(f(x - 1)$

(B) $f(g(x - 1)) > f(g(x + 1))$

(C) $g(f(x + 1)) < g(f(x - 1))$

(D) $g(g(x + 1)) < g(g(x - 1))$

54. Consider the function $f : R \to R$ defined as $f(x) = x + \sin x$. Which of the following is/are the correct statement(s)?

(A) The function is strictly increasing at every point on R except at 'x' equal to an odd integral multiple of π where the derivative of f (x) is zero and where the function f is not strictly increasing.

(B) The function is bounded in every bounded interval but unbounded on whole real line.

(C) The graph of the function y = f (x) lies in the first and third quadrants only.

(D) The graph of the function $y = f(x)$ cuts the line $y = x$ at infinitely many points.

55. For the function $f(x) = (x^2 + bx + c)e^x$, which of the following holds ?

(A) if $f(x) > 0$ for all real $x \not\Rightarrow f'(x) > 0$

(B) if $f(x) > 0$ for all real $x \Rightarrow f'(x) > 0$

(C) if $f'(x) > 0$ for all real $x \Rightarrow f(x) > 0$

(D) if $f'(x) > 0$ for all real $x \not\Rightarrow f(x) > 0$

56. The function $f(x) = (x + 2)^{1/3}$ at $x = -2$

(A) is monotonic

(B) is differentiable

(C) is such that no tangent can be drawn at this point.

(D) changes its concavity.

57. Let a function $f : R \to R$ be such that for any real numbers $a < b$, the image $f([a, b])$ is a closed interval of length $b - a$. Then

(A) f is a continuous function

(B) f is monotonous function

(C) there are only two functions $f(x) = \pm x + c$, c is a constant

(D) None of these

58. Let $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c, d are real and $3b^2 < c^2$ is an increasing function and $g(x) = af'(x) + bf''(x) + c^2$. If $G(x) = \int_{\alpha}^{x} g(t)\, dt$, $a \in R$, then

(A) G(x) is a decreasing function

(B) G(x) is an increasing function

(C) g(x) is neither increasing nor decreasing

(D) G(x) is one-one function

59. Consider a real valued continuous function $f(x)$ defined on the interval [a, b]. Which of the following statements does not hold(s) good?

(A) If $f(x) \geq 0$ on [a, b] then
$$\int_a^b f(x)\, dx \leq \int_a^b f^2(x)\, dx.$$

(B) If f(x) is increasing on [a, b], then $f^2(x)$ is increasing on [a, b].

(C) If $f(x)$ is increasing on [a, b], then $f(x) \geq 0$ on (a, b).

(D) If $f(x)$ attains a minimum at $x = c$ where $a < c < b$, then $f'(C) = 0$.

60. If $p'(x) > p(x)$ for all $x \geq 1$ and $p(1) = 0$ then

(A) $e^{-x} p(x)$ is an increasing function

(B) $p(x) . e^x$ is a decreasing function

(C) $p(x) > 0$ for all x in $[1, \infty)$

(D) $p(x) < 0$ for all x in $[1, \infty)$

61. Identify the correct statements:

(A) If $y = x + c$, then $dy = dx$.

(B) If $y = ax + b$, then $\Delta y/\Delta x = dy/dx$.

(C) If y is differentiable, then $\lim_{\Delta x \to 0} (\Delta y - dy) = 0$.

(D) If $y = f(x)$, f is increasing and differentiable, and $\Delta x > 0$, then $\Delta y \geq dy$.

62. If $h(x) = 3f\left(\frac{x^2}{3}\right) + f(3 - x^2)\ \forall\ x \in (-3, 4)$, where $f''(x) > 0\ \forall\ x \in (-3, 4)$, then h(x) is

(A) increasing in $\left(\frac{3}{2}, 4\right)$

(B) increasing in $\left(-\frac{3}{2}, 0\right)$

(C) decreasing in $\left(-3, -\frac{3}{2}\right)$

(D) decreasing in $\left(0, \frac{3}{2}\right)$

63. If $f(x) = x^3 - x^2 + 100x + 2002$, then

(A) $f(1000) > f(1001)$

(B) $f\left(\frac{1}{2000}\right) > f\left(\frac{1}{2001}\right)$

(C) $f(x - 1) > f(x - 2)$

(D) $f(2x - 3) > f(2x)$

64. If $f'(x) = g(x)(x - a)^2$ where $g(a) \neq 0$ and g is continuous at $x = a$, then

(A) f is increasing in the neighbourhood of a if $g(a) > 0$

(B) f is increasing in the neighbourhood of a if $g(a) < 0$

(C) f is decreasing in the neighbourhood of a if $g(a) > 0$

(D) f is decreasing in the neighbourhood of a if $g(a) < 0$

65. If composite function $f_1(f_2(f_3(\ldots(f_n(x)))))$ is an increasing function and if r of f_i s are decreasing function while rest are increasing, then the maximum value of $r(n - r)$ is

(A) $\frac{n^2-1}{4}$, when n is an even number

(B) $\frac{n^2}{4}$, when n is an odd number

(C) $\frac{n^2-1}{4}$, when n is an odd number

(D) $\frac{n^2}{4}$, when n is even number

66. Which of the functions have exactly one zero in the given interval

(A) $f(x) = x^3 + \frac{4}{x^2} + 7,\ (-\infty, 0)$

(B) $g(t) = \sqrt{t} + \sqrt{1+t} - 4,\ (0, \infty)$

(C) $r(\theta) = \theta + \sin^2\left(\frac{\theta}{3}\right) - 8,\ (-\infty, \infty)$

(D) $r(\theta) = \tan\theta - \cot\theta - \theta,\ (0, \pi/2)$

67. If f(x) and g(x) are two positive and increasing functions, then

(A) $(f(x))^{g(x)}$ is always increasing

(B) If $(f(x))^{g(x)}$ is decreasing when $f(x) < 1$,

(C) If $(f(x))^{g(x)}$ is increasing when $f(x) > 1$

(D) If $f(x) > 1$, then $(f(x))^{g(x)}$ is increasing

68. If $\phi(x) = 3f\left(\frac{x^2}{3}\right) + f(3 - x^2)\ \forall\ x \in (-3, 4)$ where $f''(x) > 0\ \forall\ x \in (-3, 4)$, the $\phi(x)$ is

(A) increasing in $\left(\frac{3}{2}, 4\right)$

(B) decreasing in $\left(-3, -\frac{3}{2}\right)$

(C) increasing in $\left(-\frac{3}{2}, 0\right)$

(D) decreasing in $\left(0, \frac{3}{2}\right)$

69. Let f(x) be an increasing function defined on $(0, \infty)$. If $f(2a^2 + a + 1) > f(3a^2 - 4a + 1)$, then the possible integral values of a is/are

(A) 1 (B) 2

(C) 3 (D) 4

70. Let f and g be functions from the interval $[0, \infty)$ to the interval $[0, \infty)$, f being an increasing function and g being a decreasing function, then

(A) $f\{g(x)\} \ge f\{g(0)\}$ (B) $g\{f(x)\} \le g\{f(0)\}$

(C) $f\{g(2)\} \le f(g(0))$ (D) None of these

71. If f(x) and g(x) are positive continuous function such that f(x) is an increasing function, g(x) is a monotonic function and it is given that

$$A = \int_0^{1/2} f(x).g(x)dx > \int_0^{1/2} f(x).g(1-x)dx, \text{ then}$$

(A) $f(x) < f(1-x),\ \forall\ x \in \left(0, \frac{1}{2}\right)$

(B) $g(x) < g(1-x),\ \forall\ x \in \left(0, \frac{1}{2}\right)$

(C) $f(x) > f(1-x),\ \forall\ x \in \left(\frac{1}{2}, 1\right)$

(D) $g(x) > g(1-x),\ \forall\ x \in \left(0, \frac{1}{2}\right)$

Assertion (A) and Reason (R)

(A) Both A and R are true and R is the correct explanation of A.

(B) Both A and R are true but R is not the correct explanation of A.

(C) A is true, R is false.

(D) A is false, R is true.

72. **Assertion (A) :** Both f(x) = 2cos x + 3sin x and $g(x) = \sin^{-1}\frac{x}{\sqrt{13}} - \tan^{-1}\frac{3}{2}$ are increasing, for $x \in (0, \pi/2)$.

Reason (R) : If f(x) is increasing then its inverse is also increasing.

73. **Assertion (A) :** The function f(x) is $x^4 - 8x^3 + 22x^2 - 24x + 21$ is decreasing for $x \in (2, 3)$ and $(-\infty, 1)$.

Reason (R) : f(x) is increasing for $x \in (1, 2)$ and $(3, \infty)$ and has no point of inflection.

74. **Assertion (A) :** If $f(0) = 0$, $f'(x) = \lambda n(x + \sqrt{1+x^2})$, then f(x) is positive for all $x \in R^+$.

Reason (R) : f(x) is increasing for x > 0 and decreasing for x < 0.

75. **Assertion (A) :** The function $f(x) = \frac{ae^x + be^{-x}}{ce^x + de^{-x}}$ is increasing function of x, then bc > ad.

Reason (R) : $f'(x) > 0$ for all x.

76. **Assertion (A) :** Let $f : R \to R$ be a function such that $f(x) = x^3 + x^2 + 3x + \sin x$. Then, f is one-one.

Reason (R) : f(x) is one-one if and only if f(x) is strictly monotonous.

77. Assertion (A) : For $0 < x < \frac{\pi}{2}$,

$\cos x \sin(\tan x) < \sin(\sin x)$

Reason (R) : $\frac{\tan x}{x}$ is increasing function in $\left(0, \frac{\pi}{2}\right)$.

78. Assertion (A) : If $0 < x < \tan^{-1}\frac{\pi}{2}$, then

$\sqrt[3]{\cos(\tan x).\cos^2 \sin x} < \frac{1}{3}$ [cos (tan x) + 2 cos sin x] $\leq \cos\left[\frac{\tan x + 2\sin x}{3}\right] < \cos x$.

Reason (R) : We know that AM ≥ GM and if $x \in \left(0, \frac{\pi}{2}\right)$, tan x + 2 sin x > 3x and cos x is a decreasing function.

79. Assertion (A) : If $x \in \left[\tan^{-1}\frac{\pi}{2}, \frac{\pi}{2}\right)$ then tan (sin x) > sin (tan x).

Reason (R) : $\tan\left(\sin\left(\tan^{-1}\frac{\pi}{2}\right)\right)$

$= \tan\frac{\frac{\pi}{2}}{\sqrt{1+\frac{\pi^2}{4}}} > \tan\frac{\pi}{4}$ while sin (tan x) ≤ 1 and tan (sin x) is increasing in the given interval.

80. Assertion (A) : Let a strictly decreasing function $f : R^+ \to R^+$ satisfy $f(f(x)) = x$ for all positive x. Then $\lim\limits_{x\to\infty} f(x) = 0$.

Reason (R) : Since f is strictly decreasing and bounded below by 0, we see that $\lim\limits_{x\to\infty} f(x)$ exists and is some non-negative number.

81. Assertion (A) : $(3.14)^{\pi} > \pi^{3.14}$

Reason (R) : Let $f(x) = \frac{\ell n x}{x}$, f(x) is decreasing for x > e. Since $e < 3.14 < \pi$, $f(3.14) > f(\pi)$.

Comprehension - 1

Consider f, g and h be three real valued function defined on R.

Let f (x) = sin 3x + cos x, g (x) = cos 3x + sin x and $h(x) = f^2(x) + g^2(x)$

82. The length of a longest interval in which the function y = h (x) is increasing, is

(A) $\frac{\pi}{8}$ (B) $\frac{\pi}{4}$

(C) $\frac{\pi}{6}$ (D) $\frac{\pi}{2}$

83. The general solution of the equation h (x) = 4, is

(A) $(4n+1)\frac{\pi}{8}$ (B) $(8n+1)\frac{\pi}{8}$

(C) $(2n+1)\frac{\pi}{4}$ (D) $(7n+1)\frac{\pi}{4}$

where n ∈ I

84. The number of point(s) where the graphs of the two function, y = f (x) and y = g (x) intersects in [0, π], is

(A) 2 (B) 3

(C) 4 (D) 5

Comprehension - 2

A cylinder of base radius 1 and height x is cut into two equal parts along a plane passing through the centre of the cylinder and tangent to the two base circles. Let f(x) be the ratio of surface area of each piece to the volume of the piece.

85. The value of f(2) is

(A) $2+3\sqrt{2}$ (B) $3+\sqrt{2}$

(C) $3+2\sqrt{2}$ (D) None of these

86. The complete interval on which the function f(x) is strictly decreasing, is

(A) $(0, \infty)$ (B) $(2, 4)$

(C) $(1, \infty)$ (D) None

87. The value of $\lim\limits_{x\to\infty} f(x)$ is

(A) 2 (B) 3

(C) $\frac{2\pi}{3}$ (D) None

Comprehension - 3

Consider the cubic $f(x) = 8x^3 + 4ax^2 + 2bx + a$ where a, b ∈ R.

88. For a = 1 if y = f(x) is strictly increasing ∀ x ∈ R then the largest range of values of b is

(A) $\left(-\infty, \frac{1}{3}\right]$ (B) $\left(\frac{1}{3}, \infty\right)$

(C) $\left[\frac{1}{3}, \infty\right)$ (D) $(-\infty, \infty)$

89. For $b = 1$, if $y = f(x)$ is non-monotonic then the sum of all the integral values of $a \in [1, 100]$, is

(A) 4950 (B) 5049

(C) 5050 (D) 5047

90. If the sum of the base 2 logarithms of the roots of the cubic $f(x) = 0$ is 5 then the value of 'a' is

(A) –64 (B) –8

(C) –128 (D) –256

Comprehension - 4

Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{-2, -1, 0, 1, 2, 3, 4, 5\}$. The number of

91. Increasing function from A to B is

(A) 120 (B) 72

(C) 60 (D) 56

92. Non-decreasing functions from A to B is

(A) 216 (B) 540

(C) 792 (D) 840

93. Onto functions from A to A such that $f(i) \neq i$ for all i, is

(A) 44 (B) 120

(C) 56 (D) 76

Comprehension - 5

If $\phi(x)$ is a differentiable function satisfying $\phi'(x) + 2\phi(x) \le 1$, then it can be adjusted as $e^{2x}\phi'(x) + 2e^{2x}\phi(x) \le e^{2x}$ or $\frac{d}{dx}\left(e^{2x}\phi(x) - \frac{e^{2x}}{2}\right) \le 0$.

Here e^{2x} is called a integrating factor which helps in creating a function whose differential coefficient is given.

94. If $\phi'(x) + 2\phi(x) \le 1$ for all x then the function $f(x) = e^{2x}(2\phi(x) - 1)$

(A) is a decreasing function

(B) is a increasing function

(C) is a positive function

(D) is a negative function

95. If $P(1) = 0$ and $\frac{dP(x)}{dx} > P(x)$ for all $x \ge 1$ then

(A) $P(x) > 0 \ \forall x > 1$

(B) $P(x)$ is a constant function

(C) $P(x) < 0 \ \forall x > 1$

(D) None of these

96. If $H(x_0) = 0$ for some $x = x_0$ and $\frac{d}{dx}H(x) > 2cxH(x)$ for all $x \ge x_0$, where $c > 0$, then

(A) $H(x) = 0$ has root for $x > x_0$

(B) $H(x) = 0$ has no roots for $x > x_0$

(C) $H(x)$ is a constant function

(D) None of these

MATCH THE COLUMNS FOR JEE ADVANCED

97.

	Column - I	Column - II
(A)	Let $y = f(x)$ be given by $x = \frac{1}{1+t^2}$, $y = \frac{1}{(1+t^2)}$, $t > 0$. If f is increasing in (0, a), then the greatest value of a is	(P) 3
(B)	Given $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$, if $A - \lambda I$ is a singular matrix, then $\lambda^2 - 3\lambda - 2$ is equal to	(Q) $\frac{2}{3}$
(C)	The number of solutions of the equation $\lvert x-1 \rvert^{\log_3 x^2 - \log_x 9} = (x-1)^7$, is	(R) 1
(D)	A ladder of length 5m leaning against a wall is being pulled along the ground at 2cm/s. When the foot of the ladder is 4m away from the wall, if the top of the ladder slides down on the wall at $\frac{8}{\lambda}$ cm/s, then λ is equel to	(S) 2

98.

	Column - I	**Column - II**
(A)	When 5^{200} is divided by 8m then remainder is	(P) -1
(B)	The value of $\lim\limits_{x\to\frac{x}{2}}\sqrt{\dfrac{\tan x-\sin(\tan^{-1}(\tan x))}{\tan x+\cos^2(\tan x)}}$ is	(Q) 2
(C)	If $f'(1)=-2\sqrt{2}$ and g’ ($\sqrt{2}$) = 4, then the derivative of f(tan x) with respect to g(sec x) at $x=\dfrac{\pi}{4}$ is,	(R) 1
(D)	The length of the longest interval, in which the function $f(x)=3\sin x-4\sin^3 x$ is increasing is $\dfrac{a\pi}{6}$, then value of a is	(S) does not exist

99.

	Column-I	**Column-II**
(A)	Let $f'(x)>0\ \forall\ x\in R$ and $g(x)=f(4-x)+f(2+x)$ then g(x) increases if x belongs to the interval	(P) $(-\infty,-2)$
(B)	The equation $x^3-3x+a=0$ will have exactly one real root if a b belongs to the interval	Q) $(-\infty,-1]$
(C)	If $f(x)=\cos x+a^2x+b$ is an increasing function for all values of x, then a belongs to the interval	(R) $[0,\infty)$
(D)	If f(x) $2e^x-ae^{-x}+(2a+1)x-3$ is increasing for all values of x, then a belongs to the interval	(S) $[1,\infty)$
		(T) $(2,\infty)$

100.

	Column - I	**Column - II**
(A)	The function $f(x)=\dfrac{x}{(1+x^2)}$ decreases in the interval	(P) $(-\infty,-1)$
(B)	The function $f(x)=\tan^{-1}x-x$ decreases in the interval	(Q) $(-\infty,0)$
(C)	The function $f(x)=x-e^x+\tan\left(\dfrac{2\pi}{7}\right)$ increases in the interval	(R) $(0,\infty)$
(D)	The largest interval in which $f(x)=x^3-\ln(1+x^3)$ is non negative is	(S) $(1,\infty)$
		(T) $(-1,\infty)$

101.

	Column I	**Column II**
(A)	The value of $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{r=4n}\dfrac{\sqrt{n}}{\sqrt{r}\left(3\sqrt{r}+4\sqrt{n}\right)^2}=\dfrac{p}{q}$ in its lowest form where $p+q=$	(P) 2
(B)	No. of integral values of a for which the cubic $f(x)=x^3+ax+2$ is non monotonic and has exactly one real root.	(Q) 0
(C)	The radical centre of three circles is at the origin. The equations of two of the circles are $x^2+y^2=1$ and $x^2+y^2+4x+4y-1=0$. If the equation of third circle passes through the point (1, 1) and (–2, 1) is $x^2+y^2+2gx+2fy+c=0$ then $f-c=$	(R) 1
(D)	If $x^2+y^2+z^2-2xyz=1$, then the value of $\dfrac{dx}{\sqrt{1-x^2}}+\dfrac{dy}{\sqrt{1-y^2}}+\dfrac{dz}{\sqrt{1-z^2}}$	(S) 11

Review Exercises for JEE Advanced

1. Find the critical points of the function

$$f(x) = \frac{1}{3}\sin a \tan^3 x + (\sin a - 1)\tan x + \frac{\sqrt{a-2}}{\sqrt{8-a}}$$

2. Determine the values of the number a for which the function f has no critical point.
$f(x) = (a^2 + a - 6)\cos 2x + (a-2)x + \cos 1$

3. Let $f(x) = a.\,9^x + 8(a-1)3^x + 2(a-1)\,x\ln 3$. Find all values of a so that f(x) is an increasing function for $x \in R$.

4. Find the intervals in which $f(x) = |x+1|\,|x+2|$ increases and the intervals in which it decreases.

5. A function f(x) is given by the equation, $x^2 f'(x) + 2x\,f(x) - x + 1 = 0\ (x \neq 0)$. If $f(1) = 0$, then find the intervals of monotonocity of f.

6. Prove that

$$e^x + \sqrt{1+e^{2x}} \geq (1+x) + \sqrt{2+2x+x^2} \quad \forall\ x \in R.$$

7. Find the interval to which b may belong so that

the function $f(x) = \left(1 - \frac{\sqrt{21-4b-b^2}}{b+1}\right)x^3 + 5x + \sqrt{6}$

is increasing at every point of its domain.

8. Let $f(x) = 1 - x - x^3$. Find all real values of x satisfying the inequality, $1 - f(x) - f^3(x) > f(1-5x)$.

9. If $f(x) = 2e^x - ae^{-x} + (2a+1)\,x - 3$ monotonically increases for every $x \in R$ then find the range of values of 'a'.

10. Find a polynomial f(x) of degree 4 which increases in the intervals $(-\infty, 1)$ and $(2, 3)$ and decreases in the intervals $(1, 2)$ and $(3, \infty)$ and satisfies the condition $f(0) = 1$.

11. Determine whether the function $g(x) = \tan x - 4x$ is increasing or decreasing in the interval $-\pi/3 < x < 0$.

12. Show that, $x^3 - 3x^2 - 9x + 20$ is positive for all values of $x > 4$.

13. If $0 < x < 1$ prove that $y = x\ln x - (x^2/2) + (1/2)$ is a function such that $d^2y/dx^2 > 0$. Deduce that $x\ln x > (x^2/2) - (1/2)$.

14. Find all numbers p for each of which the least value of the quadratic trinomial $4x^2 - 4px + p^2 - 2p + 2$ on the interval $0 \leq x \leq 2$ is equal to 3.

15. Find the greatest & least values of f(x)

$$= \sin^{-1}\frac{x}{\sqrt{x^2+1}} - \ln x \text{ in } \left[\frac{1}{\sqrt{3}}, \sqrt{3}\right]$$

16. Use the function $y = (\sin x)^{\sin x}$, $0 < x < \pi$, to determine the bigger of the two $\left(\frac{1}{2}\right)^e$ and $\left(\frac{1}{e}\right)^2$

17. Prove $e^{\cos x - \sin x} < \frac{1-\sin x}{1-\cos x}$ if $0 < x < \frac{\pi}{4}$.

18. Show that the equation $\cos x = x\sin x$ has exactly one solution in the interval $\left(0, \frac{\pi}{2}\right)$.

19. Find all the values of the parameter b for each of which the function $f(x) = \sin 2x - 8(b+2)\cos x - (4b^2 + 16b + 6)x$ decreases throughout the number line and has no critical points.

20. If $x \in [0, 3]$ and the greatest interval of values of x for which $2x \leq 2\sin - \sin 2x \leq 5x$ is $[a, b]$, find the value of $\frac{3a}{b}$.

21. Identify which is greater $\frac{1+e^2}{e}$ or $\frac{1+\pi^2}{\pi}$?

22. Find the points of inflection and the intervals of concavity of the graphs of the given functions.

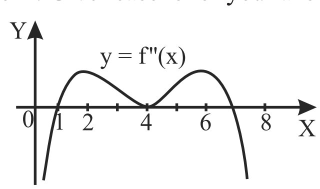
(i) $y = x^3 - 5x^2 + 3x - 5$

(ii) $y = (x+2)^6 + 2x + 2$

(iii) $y = \ln(1+x^2)$

(iv) $y = e^{\tan^{-1} x}$

23. The graph of the second derivative f'' of a function f is shown. State the x–coordinates of the inflection points of f. Give reasons for your answers.



24. Let f be a function whose second derivative is of the form $f''(x) = (x-a)^k g(x)$, where k is a positive integer, a is a fixed number, and g is a continuous function such that $g(a) \neq 0$. (a) Show that if k is odd, then a is an inflection point. (b) Show that if k is even, then a is not an inflection number.

25. Choose α and β such that the point A (2, 5/2) becomes a point of inflection of the curve $x^2y + \alpha x + \beta y = 0$. Will it have some more points of inflection? What are they?

26. Prove that $e^{\lambda\pi + (1-\lambda)e} < \lambda e^{\pi} + (1-\lambda)e^{e}$, $0 < \lambda < 1$

27. If λ & u are positive numbers whose sum is 1, using graph of $y = x^2$ prove that $(\lambda + \mu)(\lambda x_1^2 + \mu x_2^2) - (\lambda x_1 + \mu x_2)^2 \geq 0$

28. If $0 < x_1 < x_2 < x_3 < \pi$, then prove that

$$\sin\left(\frac{2x_1 + x_2 + x_3}{4}\right) > \frac{2\sin x_1 + \sin x_2 + \sin x_3}{4}$$

29. If f(x) is a monotonically increasing function $\forall x \in R$, $f''(x) > 0$ and $f^{-1}(x)$ exists, then prove that

$$\frac{f^{-1}(x_1) + f^{-1}(x_2) + f^{-1}(x_3)}{3} < f^{-1}\left(\frac{x_1 + x_2 + x_3}{3}\right).$$

30. If $f(x) \to a$ as $x \to \infty$, then prove that $f'(x)$ cannot tend to any limit other than zero.

31. If $f(x) + f'(x) \to a$ as $x \to \infty$, then prove that $f(x) \to a$ and $f'(x) \to 0$.

32. Show that $1 - \frac{1}{x} \leq \ln x \leq x - 1$ for $x > 0$.

33. (i) Show that $e^x \geq 1 + x$ for $x \geq 0$.

(ii) Deduce that $e^x \geq 1 + x + \frac{1}{2}x^2$ for $x \geq 0$

(iii) Use mathematical induction to prove that for $x \geq 0$ and any positive integer n,

$$e^x \geq 1 + x + \frac{x^2}{2!} + \ldots\ldots + \frac{x^n}{n!}.$$

34. Prove that the inequality

$$5e^{1/3} < (3x^2 - 7x + 7)\, e^x < \frac{11}{3}\sqrt[3]{e^2}.$$

is valid for $x \in [0, 2/3]$.

35. Prove that the inequality

$$\cos x\,\sqrt{\sin x} \leq 2^{\frac{1}{2}} \cdot 2^{-\frac{3}{4}}.$$

Target Exercises for JEE Advanced

1. Find the critical points of the function :

(i) $f(x) = 2\sin a \cos x + \frac{1}{3}\cos 3x + \frac{1}{\sqrt{4a - a^2}}$

(ii) $f(x) = \left(1 - \frac{\cos a}{4}\right)\sin 2x + \frac{1}{8}\sin(\pi + 4x) + x\left(\frac{\cos a - 3}{2}\right) + \sqrt{2a - a^2 + 3}$

2. Prove that $\psi(x) = \frac{1}{2}\sin x \tan x - \ln \sec x$ is positive and increasing for $0 < x < \frac{\pi}{2}$.

3. Find the intervals of monotonicity of the following functions :

(i) $f(x) = \int_{-1}^{x} (t^2 + 2t)(t^2 - 1)\, dt$

(ii) $f(x) = \int_{0}^{x} \{2\sqrt{2}\sin^2 t + (2 - \sqrt{2})\sin t - 1\}\, dt$, $x \in [0, 2\pi]$

4. Let f be defined and differentiable for every $x > 0$ and $f'(x) \to 0$ as $x \to \infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to \infty$

5. Let (i) f be continuous for $x \geq 0$

(ii) $f'(x)$ exist for $x > 0$

(iii) $f(0) = 0$

(iv) f' be strictly increasing.

If $g(x) = \frac{f(x)}{x}$, $x > 0$, prove that g is strictly increasing.

6. Let f be differentiable on [a, b], $f(a) = 0$ and there is a real number A such that $|f'(x)| \leq A|f(x)|$ on [a, b]. Prove that $f(x) = 0\ \forall x \in [a,b]$.

7. If f is a function such that $f(x) > 0$, $f'(x)$ is continuous for all x and $ax.f'(x) \geq 2\sqrt{f(x)} - 2a\, f(x)$, $(a.x \neq 2)$, show that $\sqrt{f(x)} \geq \frac{\sqrt{f(1)}}{x}$, $x \geq 1$.

8. Prove that the inequality $1 \leq \sqrt[3]{\frac{x^2}{2x-1}} \leq \sqrt[3]{\frac{4}{3}}$ holds true for $x \in [3/4, 2]$.

9. Let f be twice differentiable on $[0, \infty)$, f'' bounded on $(0, \infty)$ and let $f(x) \to 0$ as $x \to \infty$. Prove that $f'(x) \to 0$ as $x \to \infty$.

10. Prove that $\pi < \dfrac{\sin \pi x}{x(1-x)} \le 4$ when $0 < x < 1$.

11. Prove that, if $x^2 < 1$, $\tan^{-1}x$ lies between

$x - \frac{1}{3}x^2$ and $x - \frac{1}{3}x^2 + \frac{1}{5}x^3$

12. Prove that the following inequality: $\dfrac{x}{ax^2+b} \le \dfrac{1}{2\sqrt{ab}}$.

13. If $x > -1$ then prove that $x^2 > (1+x)[\ln(1+x)]^2$.

14. Show that $y = \dfrac{\ln(1+x)}{x}$ and $y = \dfrac{x}{(1+x)\ln(1+x)}$ both decrease steadily as x increases from 0 towards ∞.

15. Show that $y = \dfrac{1}{\ln(1+x)} - \dfrac{1}{x}$ decreases steadily from 1 to 0 as x increases from -1 towards ∞.

16. Show that $x/(1+x^2) < \tan^{-1}x < x$ for $x > 0$. Use this and $\dfrac{\pi}{4} = \tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{3}$ to prove that $\dfrac{14}{5} < \pi < \dfrac{10}{3}$.

17. If f(x) is continuous for $a \le x < b$, $f''(x)$ exists and $f''(x) > 0$ for $a < x < b$, then prove that $\dfrac{f(x)-f(a)}{x-a}$ strictly increases for $a < x < b$.

18. Find the range of values of 'b' so that for all real x, $f(x) = \int_0^x (bt^2 + t\sin t)\,dt$ is monotonic.

19. For decreasing function f in the interval [1, 10] we define $h(x) = f(x) - \{f(x)\}^2 + \{f(x)\}^3$. Show that $y = \max\{h(x)\}$ is a tangent to the curve $y = \frac{1}{2}[\sin x + |\sin x|]$ at infinitely many points, given that $f(1) = 1$.

20. Using the graph $x\,\ell n\,x$ for $x > 0$ prove that $a^a b^b c^c \ge \left(\dfrac{a+b+c}{3}\right)^{a+b+c}$ if a, b and c are all positive real numbers.

21. Let $g(x) = \int_a^x f(t)dt$ and f(x) satisfies $f(x+y) = f(x) + f(y) + 2xy - 1\ \forall\ x, y \in R$ and $f'(0) = \sqrt{3+a-a^2}$, then prove that g(x) is increasing.

22. The functions f(x) and g(x) are continuous for $0 \le x \le a$ and differentiable for $0 < x < a$, $f(0) = 0$, $g(0) = 0$, and $f'(x)$ and $g'(x)$ are positive. Prove that
(i) if $f'(x)$ increases with x, then $f(x)/x$ increases with x,
(ii) If $f'(x)/g'(x)$ increases with x, then $f(x)/g(x)$ increases with x. Prove that the functions

$\dfrac{x}{\sin x}, \dfrac{\frac{1}{2}x^2}{1-\cos x}, \dfrac{\frac{1}{6}x^3}{x-\sin x}, \ldots$ are strictly increasing in the interval $0 < x < \frac{1}{2}\pi$.

23. Let $p_n(x) = 1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \ldots + (-1)^n \dfrac{x^n}{n!}$ for each positive integer n.
(i) Show that $e^{-x} > p_1(x) = -1 - x$ for all $x > 0$.
(ii) Use the result of part (i) to show that $e^{-x} < p_2(x) = 1 - x + \frac{1}{2}x^2$ for all $x > 0$.
(iii) Use the result of part (ii) to show that $e^{-x} > p_3(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3$ for all $x > 0$.
(iv) Continue one step at a time in like manner until you have shown that $p_7(x) < e^{-x} < p_8(s)$ for all $x < 0$. Finally, substitute $x = 1$ in this inequality to show that $e \approx 2.718$ accurate to three decimal places.

24. (i) Let $f(x) = e^x - 1 - x$ for all x. Prove that $f'(x) \ge 0$ if $x \ge 0$ and $f'(x) \le 0$ if $x \le 0$. Use this fact to deduce the inequalities $e^x > 1 + x$, $e^{-x} > 1 - x$, valid for all $x > 0$. (When $x = 0$, these become equalities.) Integrate these inequalities to derive the following further inequalities, all valid for $x > 0$:
(ii) $e^x > 1 + x + \dfrac{x^2}{2!}$, $\quad e^{-x} < 1 - x + \dfrac{x^2}{2!}$.
(iii) $e^x > 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!}$, $e^{-x} > 1 - x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!}$.
(iv) Guess the generalization suggested and prove your result.

25. Prove that $\dfrac{1}{1+\frac{1}{x}} < \ln\left(1+\dfrac{1}{x}\right) < \dfrac{1}{x}$ if $x > 0$.

26. Prove that the derivative of the function f, defined by $f(x) = e^x(x^2 - 6x + 12) - (x^2 + 6x + 12)$, is never negative for any real value of x.

Deduce that $\{e^x(x-2)+(x+2)\}/x^2(e^x-1) < 1/6$ for all real values of x other than x = 0.

27. Prove that the inequality

$$\frac{1}{\sin(\pi/3+x)}+\frac{1}{\sin(\pi/3-x)} \geq \frac{4\sqrt{3}}{3}$$

is valid for $x \in [0, \pi/3)$

28. Show that the graph of the general cubic $y = ax^3 + 3bx^2 + 3cx + d$ is centrosymmetric about its point of inflection. (A graph is said to be centrosymmetric about a point O if for every point P of the graph there is a corresponding point P' such that the line segment PP' is bisected by O.)

29. Find the values of h, k and a that make the circle $(x - h)^2 + (y - k)^2 = a^2$ tangent to the parabola $y = x^2 + 1$ at the point (1, 2) and that also make the second derivatives d^2y/dx^2 have the same value on both curves there. Circles like this one that are tangent to a curve and have the same second derivative as the curve at the point of tangency are called osculating circles.

30. Obtain the equation of the inflectional tangents to the curve $y = (x^3 - x)/(3x^2 + 1)$.

31. Show that the curve $y = \frac{x+1}{x^2+1}$ has three points of inflection which lie on one straight line.

32. Show that the points of inflection of the curve $y = x \sin x$ lie on the curve $y^2 (4 + x^2) = 4x^2$.

33. Show that the curves $y = x^3 + x^2 - x - 1$, and $y = 2 (x^2 - x^3 + x - 1)$ touch, and cross one another at the point of contact.

34. Find all strictly monotonic functions $f : (0, \infty) \to (0, \infty)$ such that $f\left(\frac{x^2}{f(x)}\right) \equiv x$.

35. Let R be the set of real numbers. Prove that there is no function $f : R \to R$ with $f(0) > 0$, and such that $f(x + y) \geq f(x) + yf(f(x))$ for all $x, y \in R$.

Previous Year's Questions (JEE Advanced)

A. Fill in the blanks:

1. The larger of cos (ln θ) and ln (cos θ) if $e^{-\pi/2} < \theta < \pi/2$ is.................... **[IIT - 1983]**

2. The function $y = 2x^2 - \ln |x|$ is monotonically increasing for values of $x \neq 0$ satisfying the inequalities ......and monotonically decreasing for values of x satisfying the inequalities.......... **[IIT - 1983]**

3. The set of all x for which $ln(1 + x) \leq x$ is equal to.................... **[IIT - 1987]**

C. Multiple Choice Questions with ONE correct answer

4. Let f and g be increasing and decreasing functions, respectively from [0, ∞) to [0, ∞). Let h(x) = f(g(x)). If h(0) = 0, then h(x) – h(1) is **[IIT - 1987]**

(A) always zero (B) always negative
(C) always positive (D) strictly increasing

5. If $f(x) = \begin{cases} 3x^2 + 12x - 1, & -1 \leq x \leq 2 \\ 37 - x, & 2 < x \leq 3 \end{cases}$ then f(x) is **[IIT - 1993]**

(A) increasing in [–1, 2]
(B) continuous in [–1, 3]
(C) greatest at x = 2
(D) all above correct

6. The function f defined by $f(x) = (x + 2) e^{-x}$ is **[IIT - 1994]**

(A) decreasing for all x
(B) decreasing in (–∞, – 1) and increasing (–1, ∞)
(C) increasing for all x
(D) decreasing in (–1, ∞) and increasing (–∞, – 1)

7. The function $f(x) = \frac{\ln(\pi + x)}{\ln(e + x)}$ is **[IIT - 1995]**

(A) increasing on (0, ∞)
(B) decreasing on (0, ∞)
(C) increasing on (0, π/e), decreasing on (π/e, ∞)
(D) decreasing on (0, π/e), increasing on (π/e, ∞)

8. If $f(x)=\dfrac{x}{\sin x}$ and $g(x)=\dfrac{x}{\tan x}$, where $0<x\le 1$, then in this interval **[IIT - 1997]**
(A) both f(x) and g(x) are increasing functions
(B) both f(x) and g(x) are decreasing functions
(C) f(x) is an increasing function
(D) g(x) is an increasing function

9. The function $f(x)=\sin^4 x+\cos^4 x$ increases if **[IIT - 1999]**
(A) $0<x<\dfrac{\pi}{8}$ (B) $\dfrac{\pi}{4}<x<\dfrac{3\pi}{8}$
(C) $\dfrac{3\pi}{8}<x<\dfrac{5\pi}{8}$ (D) $\dfrac{5\pi}{8}<x<\dfrac{3\pi}{4}$

10. Consider the following statement S and R
S : Both sin x and cos x are decreasing function in the interval $\left(\dfrac{\pi}{2},\pi\right)$
R : If a differentiable function decreases in an interval (a, b) then its derivative also decreases in (a, b). Which of the following is true **[IIT - 2000]**
(A) both S and R wrong
(B) both S and R are correct, but R is not the correct explanation for S
(C) S is correct and R is the correct explanation for S
(D) S is correct and R is wrong

11. Let $f(x)=\int e^x (x-1)(x-2)\,dx$. Then f decreases in the interval **[IIT - 2000]**
(A) $(-\infty,-2)$ (B) $(-2,-1)$
(C) $(1,2)$ (D) $(2,\infty)$

12. If $f(x)=x\,e^{x(1-x)}$, then f(x) is **[IIT - 2001]**
(A) increasing on $[-1/2, 1]$
(B) decreasing on R
(C) increasing on R
(D) decreasing on $[-1/2, 1]$

13. The triangle formed by the tangent to the curve $f(x) = x^2 + bx - b$ at the point (1, 1) and the coordinate axes, lies in the first quadrant. If its area is 2, then the value of b is **[IIT - 2001]**
(A) -1 (B) 3
(C) -3 (D) 1

14. The length of the longest interval in which the function $3\sin x-4\sin^3 x$ is increasing, is **[IIT - 2002]**
(A) $\dfrac{\pi}{3}$ (B) $\dfrac{\pi}{2}$
(C) $\dfrac{3\pi}{2}$ (D) π

15. Let the function $g:(-\infty,\infty)\to\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$ be given by $g(u)=2\tan^{-1}(e^u)-\dfrac{\pi}{2}$. Then, g is **[IIT - 2008]**
(A) even and is strictly increasing in $(0,\infty)$
(B) odd and is strictly decreasing in $(-\infty,\infty)$
(C) odd and is strictly increasing in $(-\infty,\infty)$
(D) neither even nor odd, but is strictly increasing in $(-\infty,\infty)$

C. Comprehension

Consider the function $f(x)=1+2x+3x^2+4x^3$.
Let s be the sum of all distinct real roots of f(x) and let $t=|s|$. **[IIT - 2010]**

16. The real number s lies in the interval.
(A) $\left(-\dfrac{1}{4},0\right)$ (B) $\left(-1,-\dfrac{3}{4}\right)$
(C) $\left(-\dfrac{3}{4},-\dfrac{1}{2}\right)$ (D) $\left(0,\dfrac{1}{4}\right)$

17. The area bounded by the curve y = f(x) and the lines x = 0, y = 0 and x = t, lies in the interval :
(A) $\left(\dfrac{3}{4},3\right)$ (B) $\left(\dfrac{21}{64},\dfrac{11}{16}\right)$
(C) $(9,10)$ (D) $\left(0,\dfrac{21}{64}\right)$

18. The function $f'(x)$ is
(A) increasing in $\left(-t,\dfrac{1}{4}\right)$ and decreasing in $\left(-\dfrac{1}{4},t\right)$
(B) decreasing in $\left(-t,-\dfrac{1}{4}\right)$ and increasing in $\left(-\dfrac{1}{4},t\right)$
(C) increasing in $(-t, t)$
(D) decreasing in $(-t, t)$

D. Multiple Choice Questions with ONE OR MORE correct answers

19. Let $h(x) = f(x) - (f(x))^2 + (f(x))^3$ for every real number x. Then **[IIT - 1993]**
(A) h is increasing whenever f is increasing
(B) h is increasing whenever f is decreasing
(C) h is decreasing whenever f is decreasing
(D) nothing can be said in general.

20. For function $f(x) = x\cos\frac{1}{x}$, $x \geq 1$, **[IIT - 2009]**
(A) for atleast one x in interval $[1, \infty)$, $f(x+2) - f(x) < 2$
(B) $\lim\limits_{x\to\infty} f'(x) = 1$
(C) for all x in the interval $[1, \infty)$, $f(x+2) - f(x) > 2$
(D) $f'(x)$ is strictly decreasing in the interval $[1, \infty)$

E. Subjective Problems

21. Use the function $f(x) = x^{1/x}$, $x > 0$. to determine the bigger of the two numbers e^{π} and π^{e}. **[IIT - 1981]**

22. If $ax^2 + \frac{b}{x} \geq c$ for all positive x where $a > 0$ and $b > 0$ show that $27ab^2 \geq 4c^3$. **[IIT - 1982]**

23. Show that $1 + x\ln(x + \sqrt{x^2+1}) \geq \sqrt{1+x^2}$ for all $x \geq 0$ **[IIT - 1983]**

24. Show that $2\sin x + \tan x \geq 3x$ where $0 \leq x < \frac{\pi}{2}$. **[IIT - 1990]**

25. Let $f(x) = \begin{cases} xe^{ax}, & x \leq 0 \\ x + ax^2 - x^3, & x > 0 \end{cases}$ Where a is a positive constant. Find the interval in which $f'(x)$ is increasing. **[IIT - 1996]**

26. Using the relation $2(1-\cos x) < x^2$, $x \neq 0$ or otherwise, prove that $\sin(\tan x) > x$, for $\forall\, x \in (0, \pi/4)$ **[IIT - 2003]**

27. If $P(1) = 0$ and $\frac{dP(x)}{dx} > P(x)$ for all $x \geq 1$ then prove that $P(x) > 0$ for all $x > 1$. **[IIT - 2003]**

28. Prove that for $x \in \left[0, \frac{\pi}{2}\right]$, $\sin x + 2x \geq \frac{3x(x+1)}{\pi}$. Explain the identity if any used in the proof. **[IIT - 2003]**

ANSWERS

CONCEPT PROBLEMS—A

1. (i) neither increasing nor decreasing
(ii) strictly decreasing
(iii) strictly decreasing
(iv) strictly increasing

2. Both strictly increasing at x = a

3. strictly increasing both at x = 0 and x = 1.

4. Decreases at $x_1 = \frac{1}{2}$; increases at $x_2 = 2$ and $x_3 = e$; non-monotonous at $x_4 = 1$.

5. (i) decreases (ii) non-monotonous (decreases then increases) (iii) non-monotonous (increases then decreases)

7. strictly increasing at x = 0, 2 ; neither increasing nor decreasing at x = 1.

8. Strictly increasing

9. (i) non-monotonous (ii) strictly increasing (iii) strictly decreasing

CONCEPT PROBLEMS—B

1. (a) False (b) False

2. False

3. (i) True (ii) False

4. No **8.** increasing

9. g is discontinuous at x = 0.

10. (i) strictly increasing (ii) non-monotonous.

11. $\pi/4$ **12.** non-monotonous

13. strictly decreasing

14. Increasing **15.** Decreasing

16. $a \geq 1$ **17.** No

19. (i) $(1, \infty)$ (ii) $(-\infty, 1)$

PRACTICE PROBLEMS—A

24. decreasing

27. $-\frac{7}{2} \leq k \leq \frac{7}{2}$ **28.** $a \leq 0$

31. $(-\infty, -3)$

34. (i) $\left[\frac{1}{3}, \infty\right)$ (ii) $\left[-\infty, \frac{-4}{3}\right)$

39. $(-\infty, 3] \cup [1, \infty]$

PRACTICE PROBLEMS—B

2. $x = 1, 9$

3. (i) $0, (-1 \pm \sqrt{5})/2$ (ii) $0, \frac{8}{7}, 4$ (iii) $n\pi, n \in I$

4. (i) $\{\pm 1, 0\}$ (ii) $\{0\}$
(iii) $\{0, \ln 2\}$

5. $n/2, n \in I$

6. (i) $\{3\}$; (ii) $\{1\}$

7. (i) $\{\frac{3}{2}, \pi/2 + 2\pi n, n \in -N, -\pi/2 + 2\pi m, m \in N\}$

(ii) $\{2, (\pi/2 + 2\pi m)/6, m = 1, 0, -1, -2, \ldots;$

$\frac{1}{6}(-\pi/2 + 2\pi n), n = 3, 4, 5, \ldots\}$

8. (i) $2n\pi \pm \left(\pi - \cos^{-1}\frac{2}{3}\right), n \in I$

(ii) $\frac{n\pi}{2} + (-1)^n \frac{1}{2}\sin^{-1}\frac{a}{2}$ $-2 \le a \le 2$

9. $-6\pi, \frac{9\pi}{2}, 0$

10. $\left(-\infty, \frac{-4}{3}\right) \cup (2, \infty)$

CONCEPT PROBLEMS—C

1. (i) increases in $(-b/2a, \infty)$, decreases in $(-\infty, -b/2a)$,

(ii) increases in $(-\infty, \infty)$,

(iii) increases in $(-1, 1)$, decreases in $(-\infty, \infty)$

(iv) increases in $(-\infty, \infty)$

(v) increases in $(2\pi n - 2\pi/3, 2\pi n + 2\pi/3)$, decreases in $(2\pi n + 2\pi/3, 2\pi n + 4\pi/3), n \in I$,

(vii) increases in $\left(\frac{1}{2n+0.5}, \frac{1}{2n-0.5}\right)$, $n \in I$,

decreases in $(-\infty, -2), \left(\frac{1}{2n+0.5}, \frac{1}{2n-0.5}\right)$, $n \in I, n \ne 0$ and $(2, \infty)$

(viii) increases in $(0, 2/\ln 2)$, decreases in $(-\infty, 0)$ and $(2/\ln 2, \infty)$, (h) increases in $(0, n)$, decreases in (n, ∞).

2. (i) decrease in $(-\infty, -1)$ and $(0, 1)$, increases in $(-1, 0)$ and $(1, \infty)$;

(ii) decreases in $(-\infty, -1)$ and $(1, \infty)$ increases in $(-1, 1)$;

(iii) increases in $(-\infty, 1)$ and $\left(1, \frac{3}{2}\right)$, decreases in $\left(\frac{3}{2}, \infty\right)$.

3. $(-2 - \sqrt{3}, -1)$ and $(-1, \sqrt{3} - 2)$.

4. $(-\infty, 1/4)$

7. (i) One

PRACTICE PROBLEMS—C

9. (i) increases in $(-\infty, -1/2), (11/18, \infty)$; decreases in $(-1/2, 11/18)$

(ii) increases in $(0, 2)$; decreases in $(-\infty, 0), (2, \infty)$

(iii) increases in (e, ∞); decreases in $(0, 1), (1, e)$

(iv) increases in $\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$; increases in $\left(0, \frac{\pi}{3}\right)$, $\left(\frac{5\pi}{3}, 2\pi\right)$.

10. (i) increases in $[1, 3]$; decreases in $(-\infty, 1], (3, \infty)$

(ii) increases in $(-\infty, -2]$, $[0, \infty)$; decreases in $[-2, -1), (-1, 0]$

(iii) increases in $\left[-\frac{1}{2}, 1\right]$; decreases in $\left(-\infty, -\frac{1}{2}\right]$, $[1, \infty)$

(iv) increases for $x \in R$

11. (i) $\left[0, \frac{7\pi}{12}\right], \left[\frac{(2k+1)\pi}{4} - \frac{\pi}{6}, \frac{(2k+1)\pi}{4} + \frac{\pi}{6}\right], k < 0$,

$\left[\frac{(2m+1)\pi}{4} + \frac{\pi}{6}, \frac{(2m+3)\pi}{4} - \frac{\pi}{6}\right], m > 0$.

(ii) $\left[\frac{2}{(4k+3)\pi}, \frac{2}{(4k+1)\pi}\right]$

12. decreases in $(-\infty, 2)$ and increases in $(2, \infty)$.

13. $(-\infty, 3 - \sqrt{1 + 1/(3e)}$

14. $\{\cos 1 \cos 3, \sin 1 \sin 3\}$, it increases.

CONCEPT PROBLEMS—D

2. strictly decreasing

5. Let $a < x < b$. $f(a) \le f(x)$, $f(b) = f(a)$, hence $f(x) = f(a)$.

7. Yes. **8.** nothing can be said

PRACTICE PROBLEMS—D

21. Consider the inequality relating the expressions which are reciprocal of the left hand and right hand sides of the initial inequality.

22. $(-1, 0) \cup (0, \infty)$.

CONCEPT PROBLEMS—E

1. (i) $(-\infty, 1)$ and $(2, \infty)$, (ii) $(1, 2)$
(iii) 1, 2

3. No **7.** yes

9. $x = 0$; No **15.** No

16. $f'' \ge f'^2 \ge 0$, so $f'' \ge 0$.

PRACTICE PROBLEMS—E

17. Concave down in the neighbourhood of the point (1, 11), concave up in the neighbourhood of the point (3, 3).

19. $x = 0, \pm a\sqrt{3}$

20. $a = -2/3$, $b = -2$, $c = 7/3$

21. (i) (0, 0), (ii) (a, 0), (c) (a, b)

22. (i) Concave up in $(-2, 0)$ and $(2, \infty)$; concave down in $(-\infty, -2)$ and $(0, 2)$. Points of inflection at $(-2, 198)$, $(0, -20)$, $(2, -238)$.
(ii) Concave up in $(-\infty, -1)$ and $(1, \infty)$; concave down in $(-1, 1)$. Points of inflection at $(-1, 2e)$ and $(1, 10/e)$.

23. Concave up in $(0, \pi/4)$ and $(5\pi/4, 2\pi)$ and concave down in $(\pi/4, 5\pi/4)$.

24. (i) (0, 6), (8, 9) (ii) (6, 8)
(iii) (2, 4), (7, 9) (iv) (0, 2), (4, 7)
(v) (2, 3), (4, 9/2), (7, 4).

25. Concave down in the vicinity of the point $\left(\frac{1}{e^2}, -\frac{2}{e^4}\right)$, concave up in the neighbourhood of the point (1, 0).

26. $y = 2x$, $y = 3$

27. (1, 5/6) ; $15x - 6y = 10$.

OBJECTIVE EXERCISES

1. B	**2.** C	**3.** B
4. C	**5.** C	**6.** BC
7. C	**8.** B	**9.** A
10. C	**11.** D	**12.** B
13. B	**14.** A	**15.** C
16. C	**17.** D	**18.** D
19. B	**20.** A	**21.** B
22. C	**23.** D	**24.** A
25. C	**26.** C	**27.** A
28. A	**29.** A	**30.** C
31. D	**32.** B	**33.** C
34. B	**35.** D	**36.** A
37. B	**38.** B	**39.** D
40. D	**41.** B	**42.** A
43. C	**44.** D	**45.** C
46. D	**47.** C	**48.** A
49. C	**50.** B	**51.** BC
52. C	**53.** BC	**54.** BCD
55. AC	**56.** AD	**57.** ABC
58. BD	**59.** ABCD	**60.** AC
61. AB	**62.** ABCD	**63.** BC
64. AD	**65.** CD	**66.** ABCD
67. BD	**68.** ABCD	**69.** BCD
70. BC	**71.** ABC	**72.** A
73. A	**74.** A	**75.** D
76. C	**77.** B	**78.** B
79. A	**80.** B	**81.** A
82. B	**83.** A	**84.** C
85. B	**86.** A	**87.** B
88. C	**89.** B	**90.** D
91. D	**92.** C	**93.** A
94. A	**95.** A	**96.** B

97. (A)–(R) ; (B)–(S) ; (C)–(S) ; (D)–(P)

98. (A)–(R) ; (B)–(R) ; (C)–(P) ; (D)–(Q)

99. (A)–(ST) ; (B)–(PT) ; (C)–(PQ) ; (D)–(RST)

100. (A)–(PS) ; (B)–(PQRST) ; (C)–(Q); (D)–(T)

101. (A)–(S) ; (B)–(P) ; (C)–(Q) ; (D)–(Q)

REVIEW EXERCISES for JEE ADVANCED

1. $\{\pi n \pm \tan^{-1}\sqrt{\operatorname{cosec} a - 1}, n \in I\}$ for $a \in [2, \pi] \cup (2\pi, 8)$. For a $[\pi, 2\pi]$ the function has no critical points.

2. $-3.5 < a < -2.5$

3. If n is even, limit does not exist. If n is odd limit is 0

4. Increases in $\left(-2, -\frac{3}{2}\right)$ and $(-1, \infty)$; Decreases in $(-\infty, -2)$ and $\left(-\frac{3}{2}, -1\right)$

5. Dec in $(-\infty, 0), (1, \infty)$ incr in $(0, 1)$

7. $[-7, -1] \cup [2, 3]$

8. $(-2, 0) \cup (2, \infty)$

9. $a \geq 0$

10. $f(x) = a(x^4 - 8x^3 + 22x^2 - 24x) + 1, a \in (-\infty, 0)$

11. Decreasing

14. $a = 1 - \sqrt{2}$ or $5 + \sqrt{10}$

15. $(\pi/6) + (1/2)\, ln 3$, $(\pi/3) - (1/2)\, ln 3$.

16. $\left(\frac{1}{2}\right)^e$

19. $b < -3 - \sqrt{3}, b > -1 + \sqrt{3}$.

20. 2

21. $\frac{1+e^2}{e}$

22. (i) point of inflection $\left(\frac{5}{3}, -\frac{250}{27}\right)$; concave down in $\left(-\infty, \frac{5}{3}\right)$, concave up in $\left(\frac{5}{3}, \infty\right)$.

(ii) There is no point of inflection. The graph is concave up

(iii) points of inflection $(\pm 1, \ln 2)$; concave down in $(-\infty, -1)$ and $(1, \infty)$, concave up in $(-1, 1)$

(iv) point of inflection $\left(\frac{1}{2}, e^{\tan^{-1}\frac{1}{2}}\right)$; concave up in $(-\infty, 1/2)$, concave down in $(1/2 , \infty)$.

23. $x = 1, 7$

25. $\alpha = -20/3, \beta = 4/3$. The points $(-2, -5/2)$ and $(0, 0)$ are also points of inflection.

TARGET EXERCISES for JEE ADVANCED

1. (i) $\left\{\pi n,\ \pi n \pm \frac{1}{2}\cos^{-1}\left(-\frac{1+2\sin a}{2}\right), n \in I\right\}$ for $a \in (0, \pi/6] \cup [5\pi/6, 4)$; $\{\pi n,\ n \in I\}$ for $a \in (\pi/6, 5\pi/6)$

(ii) $\left\{\pi n,\ \pi n \pm \frac{1}{2}\cos^{-1}\frac{2-\cos a}{2}, n \in I\right\}$ for $a \in \left[-1, \frac{\pi}{2}\right]$; $\{\pi n, n \in I\}$ for $a \in \left(\frac{\pi}{2}, 3\right]$.

3. (i) f decreases in $(-2, -1)$ and $(0, 1)$ and increases in $(-\infty, -2)$, $(-1, 0)$ and $(1, \infty)$.

(ii) f decreases in $\left(0, \frac{\pi}{6}\right), \left(\frac{5\pi}{6}, \frac{5\pi}{4}\right)$ and $\left(\frac{7\pi}{4}, 2\pi\right)$ and increases in $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ and $\left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$

18. $b \geq 1$

29. $h = -4, k = \frac{9}{2}, a = \frac{5\sqrt{5}}{2}$

30. $x + y = 0, x - 2y \pm 1 = 0$.

34. $f(x) = Cx, C > 0$.

PREVIOUS YEAR'S QUESTIONS (for JEE ADVANCED)

1. $\cos(\ln\theta)$

2. increases in $\left(-\frac{1}{2}, 0\right), \left(\frac{1}{2}, \infty\right)$ and decreases in $\left(-\infty, -\frac{1}{2}\right), \left(0, \frac{1}{2}\right)$

3. $x \geq -1$

4. A

5. D

6. D

7. B

8. C

9. B

10. D

11. C

12. A

13. C

14. A

15. C

16. C

17. A

18. B

19. AC

20. BCD

21. e^{π}

25. $\left(\frac{-2}{a}, \frac{a}{3}\right)$