MATHEMATICS

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Class XII

MATRICES

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MATRICES

1. For the following pairs of matrices, determine the sum and difference, if they exist.

(a) $A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 1.5 & 6 \\ -3 & 2+i & 0 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 0 \\ 3 & -4 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -2 & 0 \end{pmatrix}$

Solution : (a) Matrices A and B are 2 × 3 and conformable for addition and subtraction.

$$A + B = \begin{pmatrix} 1+2 & -1+1.5 & 2+6 \\ 0+-3 & 1+2+i & 3+0 \end{pmatrix} = \begin{pmatrix} 3 & 0.5 & 8 \\ -3 & 3+i & 3 \end{pmatrix}$$

$$A - B = \begin{pmatrix} 1-2 & -1-1.5 & 2-6 \\ 0-(-3) & 1-(2+i) & 3-0 \end{pmatrix} = \begin{pmatrix} -1 & -2.5 & -4 \\ 3 & -1-i & 3 \end{pmatrix}$$

(b) Matrix A is 2 × 2, and B is 2 × 3. Since A and B are not the same size, they are not conformable for addition or subtraction.

2. Find the additive inverse of the matrix $A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ 3 & -1 & 2 & 2 \\ 1 & 2 & 8 & 7 \end{bmatrix}$.

Solution : The additive inverse of the 3 × 4 matrix A is the 3 × 4 matrix each of whose elements is the negative of the corresponding element of A. Therefore if we denote the additive inverse of A by –A, we have

$$-A = \begin{bmatrix} -2 & -3 & 1 & -1 \\ -3 & 1 & -2 & -2 \\ -1 & -2 & -8 & -7 \end{bmatrix}.$$

Obviously A + (–A) = (–A) + A = O, where O is the null matrix of the type 3 × 4.

3. If $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \\ 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -2 \\ 0 & 5 \\ 3 & 1 \end{bmatrix}$, find the matrix D such that A + B – D = 0.

Solution : We have A + B – D = 0

$\Rightarrow$ (A + B) + (– D) = 0 $\Rightarrow$ A + B = (– D) = D

Therefore $D = A + B = \begin{bmatrix} 0 & 2 \\ 3 & 7 \\ 5 & 6 \end{bmatrix}$.

4. If $A = \begin{bmatrix} 3 & 9 & 0 \\ 1 & 8 & -2 \\ 7 & 5 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 & 2 \\ 7 & 1 & 4 \\ 2 & 2 & 6 \end{bmatrix}$, verify that 3(A + B) = 3A + 3B.

Solution : We have $A + B = \begin{bmatrix} 3+4 & 9+0 & 0+2 \\ 1+7 & 8+1 & -2+4 \\ 7+2 & 5+7 & 4+6 \end{bmatrix} = \begin{bmatrix} 7 & 9 & 2 \\ 8 & 9 & 2 \\ 9 & 7 & 10 \end{bmatrix}$

$\therefore \quad 3(A + B) = \begin{bmatrix} 3\times 7 & 3\times 9 & 3\times 2 \\ 3\times 8 & 3\times 9 & 3\times 2 \\ 3\times 9 & 3\times 7 & 3\times 10 \end{bmatrix} = \begin{bmatrix} 21 & 27 & 6 \\ 24 & 27 & 6 \\ 27 & 21 & 30 \end{bmatrix}$.

Again $3A = 3 \begin{bmatrix} 3 & 9 & 0 \\ 1 & 8 & -2 \\ 7 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 3\times 3 & 3\times 9 & 3\times 0 \\ 3\times 1 & 3\times 8 & 3\times -2 \\ 3\times 7 & 3\times 5 & 3\times 4 \end{bmatrix} = \begin{bmatrix} 9 & 27 & 0 \\ 3 & 24 & -6 \\ 21 & 15 & 12 \end{bmatrix}$

Also $3B = 3 \begin{bmatrix} 4 & 0 & 2 \\ 7 & 1 & 4 \\ 2 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 3\times 4 & 3\times 0 & 3\times 2 \\ 3\times 7 & 3\times 1 & 3\times 4 \\ 3\times 2 & 3\times 2 & 3\times 6 \end{bmatrix} = \begin{bmatrix} 12 & 0 & 6 \\ 21 & 3 & 12 \\ 6 & 6 & 18 \end{bmatrix}$

$\therefore \quad 3A + 3B = \begin{bmatrix} 9 & 27 & 0 \\ 3 & 24 & -6 \\ 21 & 15 & 12 \end{bmatrix} + \begin{bmatrix} 12 & 0 & 6 \\ 21 & 3 & 12 \\ 6 & 6 & 18 \end{bmatrix}$

$\begin{bmatrix} 9+12 & 27+0 & 0+6 \\ 3+21 & 24+3 & -6+12 \\ 21+6 & 15+6 & 12+18 \end{bmatrix} = \begin{bmatrix} 21 & 27 & 6 \\ 24 & 27 & 6 \\ 27 & 21 & 30 \end{bmatrix} 3 \times 3.$

$\therefore$ 3 (A + B) = 3A + 3B, i.e. the scalar multiplication of matrices distributes over the addition of matrices.

5. If $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, find AB and BA and show that $AB \neq BA$.

Solution : We have $AB = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} 2.1-3.1+4.0 & 2.3+3.2+4.0 & 2.0+3.1+4.2 \\ 1.1-2.1+3.0 & 1.3+2.2+3.0 & 1.0+2.1+3.2 \\ -1.1-1.1+2.0 & -1.3+1.2+2.0 & -1.0+1.1+2.2 \end{bmatrix} = \begin{bmatrix} -1 & 12 & 11 \\ -1 & 7 & 8 \\ -2 & -1 & 5 \end{bmatrix}$

Similarly, $BA = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1.2+3.1-0.1 & 1.3+3.2+0.1 & 1.4+3.3+0.2 \\ -1.2+2.1-1.1 & -1.3+2.2+1.1 & -1.4+2.3+1.2 \\ 0.2+0.1-2.1 & 0.3+0.2+2.1 & 0.4+0.3+2.2 \end{bmatrix} = \begin{bmatrix} 5 & 9 & 13 \\ -1 & 2 & 4 \\ -2 & 2 & 4 \end{bmatrix}$

The matrix AB is of the type 3 × 3 and the matrix BA is also of the type 3 × 3. But the corresponding elements of these matrices are not equal. Hence $AB \neq BA$.

6. Show that for all values of p, q, r, s the matrices, P = $\begin{bmatrix} p & q \\ -q & p \end{bmatrix}$, and Q = $\begin{bmatrix} r & s \\ -s & r \end{bmatrix}$ commute.

Solution : We have PQ = $\begin{bmatrix} pr-qs & ps+qr \\ -qr-ps & -qs+pr \end{bmatrix}$.

Also QP = $\begin{bmatrix} r & s \\ -s & r \end{bmatrix}\begin{bmatrix} p & q \\ -q & p \end{bmatrix} = \begin{bmatrix} rp-sq & rq+sp \\ -sp-rq & -sq+rp \end{bmatrix} = \begin{bmatrix} pr-qs & ps+pq \\ -qr-ps & -qs+pr \end{bmatrix}$

for all values of p, q, r, s.

Hence PQ = QP, for all values of p, q, r, s.

7. If A = $\begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix}$ B = $\begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{bmatrix}$ and C = $\begin{bmatrix} 2 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{bmatrix}$

show that AB = AC though B $\neq$ C.

Solution : We have AB = $\begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 1.1-3.2+2.1 & 1.4+3.1-2.2 & 1.1-3.1+2.1 & 1.0-3.1+2.2 \\ 2.1+1.2-3.1 & 2.4+1.1+3.2 & 2.1+1.1-3.1 & 2.0+1.1-3.2 \\ 4.1-3.2-1.1 & 4.4-3.1+1.2 & 4.1-3.1-1.1 & 4.0-3.1-1.2 \end{bmatrix} = \begin{bmatrix} -3 & -3 & 0 & 1 \\ 1 & 15 & 0 & -5 \\ -3 & 15 & 0 & -5 \end{bmatrix}.$$

Also AC = $\begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix} \times \begin{bmatrix} 2 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & -3 & 0 & 1 \\ 1 & 15 & 0 & -5 \\ -3 & 15 & 0 & -5 \end{bmatrix}$

$\therefore$ AB = AC, though B $\neq$ C.

8. If A = $\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then $A^k = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}$, where k is any positive integer.

Solution : We shall prove the result by induction on k.

We have $A_1 = A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1+2.1 & -4.1 \\ 1 & 1-2.1 \end{bmatrix}$.

Thus the result is true when k = 1.

Now suppose that the result is true for any positive integer k i.e., $A^k = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}$ where k is any positive integer.

Now we shall show that the result is true for k + 1 if it is true for k. We have

$$A^{K+1} = AA^k = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix} = \begin{bmatrix} 3+6k-4k & -12k-4+8k \\ 1+2k-k & -4k-1+2k \end{bmatrix}$$

$$= \begin{bmatrix} 1+2+2k & -4-4k \\ 1+k & -2k-1 \end{bmatrix} = \begin{bmatrix} 1+2(k+1) & -4(1+k) \\ 1+k & 1-2(1+k) \end{bmatrix}.$$

Thus the result is true for k + 1 if it is true for k. But it is true for k = 1. Hence by induction it is true for all positive integral value of k.

9. Find real numbers c_1 and c_2 so that $I + c_1M + c_2M^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ where M = $\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ and I is the identity matrix.

Solution : $M^2 = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 9 \\ 0 & 4 \end{bmatrix}$

$I + c_1M + c_2M^2 = \begin{bmatrix} 1+c_1+c_2 & 3c_1+9c_2 \\ 0 & 1+2c_1+4c_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ $c_1 + c_2 = -1$ and $3(c_1 + c_2) + 6c_2 = 0$

$\Rightarrow$ $c_2 = 1/2,\ c_1 = -3/2$]

10. If $\begin{bmatrix} 1 & 2 & a \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 18 & 2007 \\ 0 & 1 & 36 \\ 0 & 0 & 1 \end{bmatrix}$ then find the value of (n + a). [**Ans.** 200]

Solution : Consider $\begin{bmatrix} 1 & 2 & a \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & a \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 2a+8 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & a \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 3a+24 \\ 0 & 1 & 12 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore$ $\begin{bmatrix} 1 & 2 & a \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 2n & na+8\sum_{k=0}^{n-1} k \\ 0 & 1 & 4n \\ 0 & 0 & 1 \end{bmatrix}^n$

hence n = 9 and $2007 = 9a + 8\sum_{k=0}^{8} k = 9a + 8\left(\frac{8 \cdot 9}{2}\right)$

$2007 = 9a + 32 \cdot 9 = 9(a + 32)$ $a + 32 = 223$ $\Rightarrow$ $a = 191$ hence $a + n = 200$

11. Find the matrices of transformations T_1T_2 and T_2T_1, when T_1 is rotation through an angle 60º and T_2 is the reflection in the y–axis. Also verify that $T_1T_2 \neq T_2T_1$.

Solution : $T_1 = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$

and $T_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ $\therefore$ $T_1T_2 = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \times \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1+0 & 0-\sqrt{3} \\ -\sqrt{3}+0 & 0+1 \end{bmatrix}$

$= \begin{bmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$ ....(1)

and $T_2T_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \times \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1+0 & \sqrt{3}+0 \\ 0+\sqrt{3} & 0+1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$

$= \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$ .... (2)

It is clear from (1) and (2), $T_1T_2 \neq T_2T_1$

12. Find the possible square roots of the two rowed unit matrix I.

Solution : Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be square root of the matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $A^2 = I$.

i.e. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ i.e. $\begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & cb+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Since the above matrices are equal, therefore

$a^2 + bc = 1$ ...(i) $ac + cd = 0$ ...(iii)

$ab + bd = 0$ ...(ii) $cb + d^2 = 0$ ...(iv)

must hold simultaneously.

If a + d = 0, the above four equations hold simultaneously if d = –a and $a^2 + bc = 1$.

Hence one possible square root of I is

$A = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ where α, β, γ are any three numbers related by the condition $\alpha^2 + \beta\gamma = 1$.

If $a + d \neq 0$, the above four equations hold simultaneously if b = 0, c = 0, a = 1, d = 1 or if b = 0, c = 0, a = –1, d = –1.

Hence $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

i.e. ±I are other possible square roots of I.

13. Show that the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent and find its index.

Solution : We have $A^2 = AA \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$

Again $A^3 = AA^2 = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$.

Thus 3 is the least positive integer such that $A^3 = 0$. Hence the matrix A is nilpotent of index 3.

14. If AB = A and BA = B then $B'A' = A'$ and $A'B' = B'$ and hence prove that A′ and B′ are idempotent.

Solution : We have $AB = A \Rightarrow (AB)' = A' \Rightarrow B'A' = A'$.

Also $BA = B \Rightarrow (BA)' = B' \Rightarrow A'B' = B'$.

Now A′ is idempotent if $A'^2 = A'$. We have

$A'^2 = A'A' = A'(B'A') = (A'B')A' = B'A' = A'$.

∴ A′ is idempotent.

Again $B'^2 = B'B' = B'(A'B') = (B'A')B' = A'B' = B'$.

∴ B′ is idempotent.

15. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n = (a_{ij}(n))$. If $\lim_{n\to\infty} \frac{a_{12}(n)}{a_{22}(n)} = l$ where $l^2 = \sqrt{a} + \sqrt{b}$ (a, b ∈ N), find the value of (a + b). [Ans. 17]

Solution : Suppose $A_1 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = I + B$ (say)

hence $A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n = (I + B)^n$

$\therefore \quad A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n = {}^nC_0 I + {}^nC_1 B + {}^nC_2 B^2 + {}^nC_3 B^3 + {}^nC_4 B^4 + \ldots\ldots \quad \ldots.(1)$

now $\quad B^2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I$

Hence $B^{2k} = 2^k I$ and $B^{2k+1} = B^{2k}B = 2^k B$

now $\quad \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n = \underbrace{({}^nC_0 + {}^nC_2 \cdot 2 + {}^nC_4 \cdot 2^2 + \ldots\ldots)}_{\text{'X' say}} I + \underbrace{({}^nC_1 + {}^nC_3 \cdot 2 + {}^nC_5 \cdot 2^2 + \ldots\ldots)}_{\text{'Y' say}} B$

$\therefore \quad \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} + \begin{bmatrix} Y & Y \\ Y & -Y \end{bmatrix} = \begin{bmatrix} X+Y & Y \\ Y & X-Y \end{bmatrix}$

Hence a_{12} in $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n = Y \qquad \therefore \quad a_{12} = {}^nC_1 + {}^nC_3 \cdot 2 + {}^nC_5 \cdot 2^2 + {}^nC_7 \cdot 2^3 + \ldots\ldots$

$= \frac{1}{\sqrt{2}}\left[{}^nC_1 \cdot \sqrt{2} + {}^nC_3 \cdot (\sqrt{2})^3 + {}^nC_5 \cdot (\sqrt{2})^5 + \ldots\ldots\right] = \frac{1}{\sqrt{2}}\left[\frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2}\right]$

|||ly $\quad a_{22} = X - Y$

$= ({}^nC_0 + {}^nC_2 \cdot 2 + {}^nC_4 \cdot 2^2 + {}^nC_6 \cdot 2^3 + \ldots\ldots) - ({}^nC_1 + {}^nC_3 \cdot 2 + {}^nC_5 \cdot 2^2 + {}^nC_7 \cdot 2^3 + \ldots..)$

$= \frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2} - \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}$

$= \frac{\sqrt{2}[(1+\sqrt{2})^n + (1-\sqrt{2})^n] - [(1+\sqrt{2})^n - (1-\sqrt{2})^n]}{2\sqrt{2}}$

$a_{22} = \frac{(\sqrt{2}-1)(1+\sqrt{2})^n - (\sqrt{2}+1)(1-\sqrt{2})^n}{2\sqrt{2}}$

$\therefore \quad \lim_{n\to\infty} \frac{a_{12}}{a_{22}} = \lim_{n\to\infty} \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{(\sqrt{2}-1)(1+\sqrt{2})^n + (\sqrt{2}+1)(1-\sqrt{2})^n}$

$= \lim_{n\to\infty} \frac{1 - \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^n}{(\sqrt{2}-1) + (\sqrt{2}+1)\left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^n} = \frac{1-0}{\sqrt{2}-1} = 1+\sqrt{2}\,;$

hence $l^2 = (1+\sqrt{2})^2 = 3 + 2\sqrt{2} = \sqrt{9} + \sqrt{8}$. Hence a + b = 9 + 8 = 17 Ans.

16. Prove that $\Delta \equiv \begin{vmatrix} 1 & bc+ad & b^2c^2+a^2d^2 \\ 1 & ca+bd & c^2a^2+b^2d^2 \\ 1 & ab+cd & a^2b^2+c^2d^2 \end{vmatrix} = (a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$.

Solution : Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$\Delta = \begin{vmatrix} 1 & bc+ad & b^2c^2+a^2d^2 \\ 1 & (a-b)(c-d) & (a^2-b^2)(c^2-d^2) \\ 1 & (a-c)(b-d) & (a^2-c^2)(b^2-d^2) \end{vmatrix} = \begin{vmatrix} (a-b)(c-d) & (a-b)(a+b)(c-d)(c+d) \\ (a-c)(b-d) & (a-c)(a+c)(b-d)(b+d) \end{vmatrix}$

$=(a-b)(c-d)(a-c)(b-d) \begin{vmatrix} 1 & (a+b)(c+d) \\ 1 & (a+c)(b+d) \end{vmatrix}$

$= (a-b)(c-d)(a-c)(b-d)[(a+c)(b+d) - (a+b)(c+d)]$

$= (a-b)(c-d)(a-c)(b-d)(ab+cd-ac-bd)$

$= (a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$.

17. Show that , $\begin{vmatrix} yz-x^2 & zx-y^2 & xy-y^2 \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix} = \begin{vmatrix} r^2 & u^2 & u^2 \\ u^2 & r^2 & u^2 \\ u^2 & u^2 & r^2 \end{vmatrix}$

where $r^2 = x^2 + y^2 + z^2$ & $u^2 = xy + yz + zx$.

Solution : Consider the determinant , $\Delta = \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}$

We see that the L.H.S. determinant has its constituents which are the co-factor of Δ . Hence L.H.S. determinant

$$= \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix} \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix} = \begin{vmatrix} x^2+y^2+z^2 & xy+yz+zx & xy+yz+zx \\ xy+yz+zx & y^2+z^2+x^2 & yz+zx+xy \\ zx+xy+yz & yz+xz+xy & z^2+x^2+y^2 \end{vmatrix} = \begin{vmatrix} r^2 & u^2 & u^2 \\ u^2 & r^2 & u^2 \\ u^2 & u^2 & r^2 \end{vmatrix}$$

18. Without expanding, as for as possible, prove that

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)$$

Solution : Let D = $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix}$ for x = y, D = 0 (since C_1 and C_2 are identical)

Hence (x – y) is a factor of D (y – z) and (z – x) are factors of D. But D is a homogeneous expression of the 4th degree is x, y, z.

∴ There must be one more fator of the 1st degree in x, y, z say k (x + y + z) where k is a constant.

Let D = k (x – y) (y – z) (z – x) (x + y + z)

Putting x = 0, y = 1, z = 2

then $\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 8 \end{vmatrix} = k(0-1)(1-2)(2-0)(0+1+2)$

⇒ L(8 – 2) = k(–1) (–1) (2) (3) ∴ k = 1 ∴ D = (x – y) (y – z) (z – x) (x + y + z)

19. Express $\begin{vmatrix} (1+ax)^2 & (1+ay)^2 & (1+az)^2 \\ (1+bx)^2 & (1+by)^2 & (1+bz)^2 \\ (1+cx)^2 & (1+cy)^2 & (1+cz)^2 \end{vmatrix}$ as product of two determinatnts.

Solution : The given determinant is = $\begin{vmatrix} 1+2ax+a^2x^2 & 1+2ay+a^2y^2 & 1+2az+a^2z^2 \\ 1+2bx+b^2x^2 & 1+2by+b^2y^2 & 1+2bz+b^2z^2 \\ 1+2cx+c^2x^2 & 1+2cy+c^2y^2 & 1+2cz+c^2z^2 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 2a & a^2 \\ 1 & 2b & b^2 \\ 1 & 2c & c^2 \end{vmatrix} \times \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$, with the help of row–by–row multiplication rule.

20. Let D = $\begin{vmatrix} 2a_1b_1 & a_1b_2+a_2b_1 & a_1b_3+a_3b_1 \\ a_1b_2+a_2b_1 & 2a_2b_2 & a_2b_3+a_3b_2 \\ a_1b_3+a_3b_1 & a_3b_2+a_2b_3 & 2a_3b_3 \end{vmatrix}$.

Express the determinant D as a product of two determinants. Hence or otherwise show that D = 0.

Solution : We have $D = \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} \times \begin{vmatrix} b_1 & a_1 & 0 \\ b_2 & a_2 & 0 \\ b_3 & a_3 & 0 \end{vmatrix}$, as can be seen by applying row–by–row multiplication rule. Hence D = 0.

21. If $f(x, y) = x^2 + y^2 - 2xy$, $(x, y \in R)$ and the matrix A is given by $A = \begin{bmatrix} f(x_1, y_1) & f(x_1, y_2) & f(x_1, y_3) \\ f(x_2, y_1) & f(x_2, y_2) & f(x_2, y_3) \\ f(x_3, y_1) & f(x_3, y_2) & f(x_3, y_3) \end{bmatrix}$ such that trace (A) = 0, then prove that det. (A) ≥ 0.

Solution : tr (A) = 0 $\Rightarrow$ $(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 = 0$

$\Rightarrow$ $x_1 = y_1$, $x_2 = y_2$, $x_3 = y_3$

$$|A| = \begin{vmatrix} x_1^2 - 2x_1y_1 + y_1^2 & x_1^2 - 2x_1y_2 + y_2^2 & x_1^2 - 2x_1y_3 + y_3^2 \\ x_2^2 - 2x_2y_1 + y_1^2 & x_2^2 - 2x_2y_2 + y_2^2 & x_2^2 - 2x_2y_3 + y_3^2 \\ x_3^2 - 2x_3y_1 + y_1^2 & x_3^2 - 2x_3y_2 + y_2^2 & x_3^2 - 2x_3y_3 + y_3^2 \end{vmatrix} = \begin{vmatrix} x_1^2 & -2x_1 & 1 \\ x_2^2 & -2x_2 & 1 \\ x_3^2 & -2x_3 & 1 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ y_1^2 & y_2^2 & y_3^2 \end{vmatrix}$$

$$= 2(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)(y_1 - y_2)(y_2 - y_3)(y_3 - y_1) \quad = 2\left((x_1 - x_2)(x_2 - x_3)(x_3 - x_1)\right)^2 \geq 0$$

Alternatively: $$|A| = \begin{vmatrix} x_1^2 - 2x_1y_1 + y_1^2 & x_1^2 - 2x_1y_2 + y_2^2 & x_1^2 - 2x_1y_3 + y_3^2 \\ x_2^2 - 2x_2y_1 + y_1^2 & x_2^2 - 2x_2y_2 + y_2^2 & x_2^2 - 2x_2y_3 + y_3^2 \\ x_3^2 - 2x_3y_1 + y_1^2 & x_3^2 - 2x_3y_2 + y_2^2 & x_3^2 - 2x_3y_3 + y_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & (x_1 - x_2)^2 & (x_1 - x_3)^2 \\ (x_2 - x_1)^2 & 0 & (x_2 - x_3)^2 \\ (x_3 - x_1)^2 & (x_3 - x_2)^2 & 0 \end{vmatrix}$$

22. Find the adjoint of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$.

Solution : We have $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{vmatrix}$.

The cofactors of the elements of the first row of | A | are $\begin{vmatrix} 5 & 0 \\ 4 & 3 \end{vmatrix}, -\begin{vmatrix} 0 & 0 \\ 2 & 3 \end{vmatrix}, \begin{vmatrix} 0 & 5 \\ 2 & 4 \end{vmatrix}$ i.e., are 15, 0, – 10 respectively.

The cofactors of the elements of the second row of |A| are $-\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix}, \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix}, -\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$ i.e. are 6, –3, 0 respectively.

The cofactors of the elements of the third row of |A| are $\begin{vmatrix} 2 & 3 \\ 5 & 0 \end{vmatrix}, -\begin{vmatrix} 1 & 3 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix}$ i.e., are – 15, 0, 5 respectively.

Therefore the adj. A = the transpose of the matrix B where

$$B = \begin{bmatrix} 15 & 0 & -10 \\ 6 & -3 & 0 \\ -15 & 0 & 5 \end{bmatrix}. \qquad \therefore \qquad \text{adj } A = \begin{bmatrix} 15 & 0 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}.$$

23. If A and B are square matrices of the same order, then adj (AB) = adj B. adj A.

Solution : We have AB adj (AB) = $|AB|\, I_n$ = (adj AB) AB. ...(1)

Also AB (adj B. adj A) = A(B adj B) adj A

= A $|B|\, I_n$ adj A = $|B|$ (A adj A)

= $|B|\,|A|\, I_n = |BA|\, I_n$ = $|AB|\, I_n$. ...(2)

Similarly, we have

(adj B adj A) AB = adj B [(adj A [(adj A) A] B

= adj B. $|A|\, I_n$ B = $|A|$. (adj B) B

= $|A|\,.\,|B|\, I_n = |AB|\, I_n$. ...(3)

From (1), (2) and (3), the required result follows, provided AB is non–singular.

Note : The result adj (AB) = adj B adj A holds goods even if A or B is singular. However the proof given above is applicable only if A and B are non–singular.

24. If(l_r, m_r, n_r), r = 1, 2, 3 be the direction cosines of three mutually perpendicular lines referred to an orthogonal cartesian co–ordinate system, then prove that

$\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$ is an orthogonal matrix.

Solution : Let $A = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$.

Then $A' = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$. We have $AA' = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$.

$$= \begin{bmatrix} l_1^2 + m_1^2 + n_1^2 & l_1 l_2 + m_1 m_2 + n_1 n_2 & l_1 l_3 + m_1 m_2 + n_1 n_3 \\ l_2 l_1 + m_2 m_1 + n_2 n_1 & l_2^2 + m_2^2 + n_2^2 & l_2 l_3 + m_2 m_3 + n_2 m_3 \\ l_3 l_1 + m_3 m_1 + n_3 n_1 & l_3 n_2 + m_3 m_2 + n_3 n_2 & l_3^2 + m_3^2 + n_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.$$

$\begin{bmatrix} \therefore & l_1^2 + m_1^2 + n_1^2 = 1 \text{ etc.} \\ \text{and} & l_1 l_2 + m_1 m_2 + n_2 n_3 = 0 \text{ etc.} \end{bmatrix}$ Hence the matrix A is orthogonal.

25. Obtian the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ and verify that it is satisfied y A and hence find its inverse.

Solution : We have $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix}$

$= (1 - \lambda)(2 - \lambda)(3 - \lambda) + 2[0 - 2(2 - \lambda)] \quad = (2 - \lambda)[(1 - \lambda)(3 - \lambda) - 4]$

$= (2 - \lambda)[\lambda^2 - 4\lambda - 1] \quad = -(\lambda^3 - 6\lambda^2 + 7\lambda + 2).$

$\therefore$ the characteristic equaion of A is $\lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0$ ......(i)

By the Cayley–Hamilton theorem $A^3 - 6A^2 + 7A + 2I = O.$ .....(ii)

Verification of (ii). We have

$$A^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}.$$

Also $A^3 = A \,.\, A^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \times \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}.$

Now $A^2 - 6A^2 + 7A + 2I$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6 \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{O}.$$

Hence Cayley–Hamilton theorem is verified. Now we shall compute A^{-1}.

Multiplying (ii) by A^{-1}, we get $A^2 - 6A + 7I + 2A^{-1} = \mathbf{O}$.

$$\therefore \quad A^{-1} = -\frac{1}{2}(A^2 - 6A + 7I) = -\frac{1}{2} \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 3 & 13 \end{bmatrix} + 3 \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} - \frac{7}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 2 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 2 & 0 & -1 \end{bmatrix}.$$

26. Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$.

Solution : We have $|A| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 2 & -1 \\ 3 & 1 & -1 \end{vmatrix}$, aplying $C_3 \to C_3 \to 2C_2 = -1 \begin{vmatrix} 1 & -1 \\ 3 & -1 \end{vmatrix}$,

expanding the determinant along the first row = –2.

Since $|A| \neq 0$, therefore A^{-1} exists.

Now the cofactors of the elements of the first row of |A| are $\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}, |\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$ i.e., are –1, 8, –5 respectively.

The cofactors of the elements of the second row of |A| are $-\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix}, -\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix}$ i.e. are 1, –6, 3 respectively.

The cofactors of the elements of the third row of |A| are $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}, -\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}$ i.e. are –1, 2, –1 respectively.

Therefore the Adj. A = the transpose of the matrix B where

$B = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix}$. $\therefore$ Adj. $A = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ 5 & 3 & -1 \end{bmatrix}$.

Now $A^{-1} = \frac{1}{|A|}$ Adj. A and here $|A| = -2$.

$$\therefore \quad A^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}.$$

27. Sove the equations

$\lambda x + 2y - 2z - 1 = 0,$

$4x + 2\lambda y - z - 2 = 0,$

$6x + 6y + \lambda z - 3 = 0,$ considering specially the case when $\lambda = 2$.

Solution : The matrix form of the given system is $\begin{bmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ .....(i)

The given system of equations will have a unique solution if and only if the coefficient matrix is non–singular, i.e., iff

$$\begin{vmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{vmatrix} \neq 0 \quad \text{i.e., iff} \quad \lambda^3 + 11\lambda - 30 \neq 0$$

i.e., iff $(\lambda - 2)(\lambda^2 + 2\lambda + 15) \neq 0$.

Now the only real root of the equation $(\lambda - 2)(\lambda^2 + 2\lambda + 15) \neq 2 = 0$ is $\lambda = 2$

Therefore if $\lambda \neq 2$, the given system of equations will have a unique solution given by

$$\frac{x}{\begin{vmatrix} 1 & 2 & -2 \\ 2 & 2\lambda & -1 \\ 3 & 6 & \lambda \end{vmatrix}} = \frac{y}{\begin{vmatrix} \lambda & 1 & -2 \\ 4 & 2 & -1 \\ 6 & 3 & \lambda \end{vmatrix}} = \frac{z}{\begin{vmatrix} \lambda & 2 & 1 \\ 4 & 2\lambda & 2 \\ 6 & 6 & 3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{vmatrix}}$$

In case $\lambda = 2$, the equation (i) becomes $\begin{bmatrix} 2 & 2 & -2 \\ 4 & 4 & -1 \\ 6 & 6 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Performing $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, we get $\begin{bmatrix} 2 & 2 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

The above system of equations is equivalent to $8z = 0$, $3z = 0$, $2x + 2y - 2z = 1$.

$\therefore$ $x = \frac{1}{2} - c$, $y = c$, $z = 0$ consititute the general solution of the given system of equations in case $\lambda = 2$.

28. Solve

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 4x_1 + 5x_2 + 6x_3 &= 7 \\ 7x_1 + 8x_2 + 9x_3 &= 10 \end{aligned}$$

Solution :

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 4x_1 + 5x_2 + 6x_3 &= 7 \\ 7x_1 + 8x_2 + 9x_3 &= 10 \end{aligned} \xrightarrow[-7E1+E3]{04E1+E2} \begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ -3x_2 - 6x_3 &= -9 \\ -6x_2 - 12x_3 &= -18 \end{aligned}$$

$$\xrightarrow{-2E_2+E3} \begin{aligned} x_1+2x_2+3x_3 &= \ \ 4 \\ -3x_2-6x_3 &= -9 \\ 0 &= \ \ 0 \end{aligned} \qquad \xrightarrow{-\frac{1}{3}E_2} \begin{aligned} x_1+2x_2+3x_3 &= 4 \\ x_2-2x_3 &= 3 \\ 0 &= 0 \end{aligned}$$

Now we have only two equations in three unknowns. In the second equation, we can let $x_3 = k$, where k is any complex number. Then $x_2 = 3 - 2k$. Substituting $s_3 = k$ and $x_2 = 3 - 2k$ into the first eqution, we have

$$x_1 = 4 - 2x_2 - 3x_3 = 4 - 2(3 - 2k) - 3(k) = -2 + k$$

Thus the general solution is

$$(-2 + k, 3 - 2k, k) \quad \text{or} \quad \begin{aligned} x_1 &= -2+k \\ x_2 &= 3-2k \\ x_3 &= k \end{aligned}$$

And we see that the system has an infinite number of solutions. Specific solutions can be generated by choosing specific values for k.

29. Find the rank of the matrix $A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$.

Solution : We have $A \sim \begin{bmatrix} 4 & 2 & 1 & 0 \\ 6 & 3 & 4 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}$ by $C_4 \to C_4 - C_2 - C_2$

$\sim \begin{bmatrix} 0 & 0 & 1 & 0 \\ -10 & -5 & 4 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}$ by $C_2 \to C_2 - 2C_3$, $C_1 \to C_1 - 4C_3$.

We see that each minor of order 3 in the last equivalent matrix is equal to zero. But there is a minor of order 2 i.e., $\begin{vmatrix} -5 & 4 \\ 1 & 0 \end{vmatrix}$ which is equla to –4 i.e. , ≠ 0. Here rank A = 2.

30. Find the rank of the martix $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

Solution : We have the matrix $A \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ by $R_4 \to R_4 \to R_3 - R_2 - R_1$

or $A \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ by $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$.

Note E–transformations do not change the rank of a matrix. We see that the determinant of the last equivalent matrix is zero. But the leading minor of the third order of this matrix

i.e. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -3 \\ 0 & -4 & -8 \end{bmatrix}$ = –12 i.e. $\neq 0$. Therefore the rank of this matrix is 3. Hence rank A = 3.

31. Number of triplets of a, b & c for which the system of equations,

$$ax - by = 2a - b \quad \text{and} \quad (c + 1)x + cy = 10 - a + 3b$$

has infinitely many solutions and x = 1, y = 3 is one of the solutions, is :

(A) exactly one (B) exactly two

(C) exactly three (D) infinitely many

Solution : put x = 1 & y = 3 in 1st equation $\Rightarrow$ a = – 2b & from 2nd equation

$c = \frac{9 + 5b}{4}$; Now use $\frac{a}{c+1} = -\frac{b}{c} = \frac{2a - b}{10 - a + 3b}$; from first two b = 0 or c = 1 ;

if b = 0 $\Rightarrow$ a = 0 & c = 9/4 ; if c = 1 ; b = – 1 ; a = 2

32. Solve
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 4x_1 + 5x_2 + 6x_3 &= 7 \\ 7x_1 + 8x_2 + 9x_3 &= 12 \end{aligned}$$

Solution :
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 4x_1 + 5x_2 + 6x_3 &= 7 \\ 7x_1 + 8x_2 + 9x_3 &= 12 \end{aligned}$$
$$\Downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & 12 \end{array}\right) \xrightarrow[-7E2+E3]{-4E1+E2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -6 & -12 & -16 \end{array}\right)$$

$$\xrightarrow{-2E2+E3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & 0 & 0 & 2 \end{array}\right) \qquad \begin{aligned} x_1 + 2x_2 + 3x_2 &= 4 \\ 0x_1 - 3x_2 - 6x_3 &= -9 \\ 0x_1 + 0x_2 + 0x_3 &= 2 \end{aligned}$$

The last equation, 0 = 2, can never hold regardless of the values assigned to x_1, x_2 and x_3. Because the last (equivalent) system has no solution, the original system of equations has no solution.

33. Solve
$$\begin{aligned} x_2 - x_3 &= -9 \\ 2x_1 - x_2 + 4x_3 &= 29 \\ x_1 + x_2 - 3x_3 &= -20 \end{aligned}$$

by reducing the augmented matrix of the system to reduced row echelon form.

Solution :
$$\left(\begin{array}{ccc|c} 0 & 1 & -1 & -9 \\ 2 & -1 & 4 & 29 \\ 1 & 1 & -3 & -20 \end{array}\right) \xrightarrow{R1 \leftrightarrow R3} \left(\begin{array}{ccc|c} 1 & 1 & -3 & -20 \\ 2 & -1 & 4 & 29 \\ 0 & 1 & -1 & -9 \end{array}\right)$$

$$\xrightarrow{-2R1+R2} \left(\begin{array}{ccc|c} 1 & 1 & -3 & -20 \\ 0 & -3 & 10 & 69 \\ 0 & 1 & -1 & -9 \end{array}\right) \xrightarrow{-\frac{1}{3}R2} \left(\begin{array}{ccc|c} 1 & 1 & -3 & -20 \\ 0 & 1 & -\frac{10}{3} & -23 \\ 0 & 1 & -1 & -9 \end{array}\right)$$

$$\xrightarrow{-1R2+R3} \left(\begin{array}{ccc|c} 1 & 1 & -3 & -20 \\ 0 & 1 & -\frac{10}{3} & -23 \\ 0 & 0 & \frac{7}{3} & 14 \end{array}\right) \xrightarrow{\frac{3}{7}R3} \left(\begin{array}{ccc|c} 1 & 1 & -3 & -20 \\ 0 & 1 & -\frac{10}{3} & -23 \\ 0 & 0 & 1 & 6 \end{array}\right)$$

$$\xrightarrow[\substack{3R3+R1 \\ -1R2+R1}]{\frac{10}{3}R3+R2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 6 \end{array}\right)$$

It is easy to see that $x_1 = 1$, $x_2 = -3$, $x_3 = 6$.
The process of solving a system by reducing the augmented matrix to reduced row echelon form is called Gauss–Jordan elimination.

34. Solve completely the system of equations
$x + y + z = 0$, $2x - y - 3z = 0$, $3x - 5y + 4z = 0$, $x + 17y + 4z = 0$,
Solution : The given system of equations is equivalent to the single matrix equation

$$AX = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -3 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = O.$$

We shall first find the rank of the coefficient matrix A by reducing it to Echelon form by applying elementary row transformations only. Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - R_1$, we get

$$A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -5 \\ 0 & -8 & 1 \\ 0 & 16 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -5 \\ 0 & -24 & 3 \\ 0 & 48 & 9 \end{bmatrix} \text{ by } R_3 \to 3R_3,\ R_4 \to 3R_4$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -5 \\ 0 & 0 & 43 \\ 0 & 0 & -71 \end{bmatrix} \text{ by } R_3 \to R_3 - 8R_2,\ R_4 \to R_4 + 16R_2. \quad \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -5 \\ 0 & 0 & 43 \\ 0 & 0 & 0 \end{bmatrix} \text{ by } R_4 \to R_4 + \frac{71}{43} R_3.$$

Above is the Echelon form of the coefficient matrix A. We have rank A = the number of non zero rows in this Echelon form =3. The number of unknowns is also 3. Sice rank A is equal to the number of unknowns, therefore the given system of equations possesses no non–zero solution. Hence the zero solution i.e. $x = y = z = 0$ is the only solution of the given system of equations.

35. Solve completely the system of equations
$4x + 2y + z + 3u = 0,$
$6x + 3y + 4z + 7u = 0,$
$2x + y + u = 0.$

Solution : The matrix form of the given system is $\begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \mathbf{O}$

or $\begin{bmatrix} 1 & 2 & 4 & 3 \\ 4 & 3 & 6 & 7 \\ 0 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} z \\ y \\ x \\ u \end{bmatrix} = O$, interchanging the variables x and z.

Performing $R_2 \to R_2 - 4R_1$, we get $\begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & -5 & -10 & -5 \\ 0 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} z \\ y \\ x \\ u \end{bmatrix} = \mathbf{O}$.

Performing $R_2 \to -\frac{1}{5} R_2$, we get $\begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} z \\ y \\ x \\ u \end{bmatrix} = \mathbf{O}$.

Performing $R_3 \to R_3 - R_2$, we get $\begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z \\ y \\ x \\ u \end{bmatrix} = \mathbf{O}$.

The coefficient matrix is of rank 2 and therefore the given system will have 4 – 2 i.e. 2 linearly independent solutions. The given system of equations is equivalent to

$z + 2y + 4x + 3u = 0,$

$y + 2x + u = 0.$

$\therefore$ $y = -2x - u$, $z = -4x - 3u + 4x + 2u = -u$.

$\therefore$ $x = c_1$, $u = c_2$, $y = -2c_1 - c_2$, $z = c_2$

consitute the general solution where c_1 and c_2 are arbitrary constats.

36. Determine conditions on a, b and c so that

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= a \\ 4x_1 + 5x_2 + 6x_3 &= b \\ 7x_1 + 8x_2 + 9x_3 &= c \end{aligned}$$

will have no solutions or have an infinite number of solution.

Solution : $\left(\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & -3 & -6 & b-4a \\ 0 & 0 & 0 & c-2b+a \end{array}\right)$

If $c - 2b + a \neq 0$, then no solution exists. If $c - 2b + a = 0$, we have two equations in three unknowns and we can set x_3 arbitrarily and then solve for x_1 and x_2.

37. Solve

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 4x_1 + 5x_2 + 6x_3 &= 0 \\ 7x_1 + 8x_2 + 9x_3 &= 0 \\ 10x_1 + 11x_2 + 12x_3 &= 0 \end{aligned}$$

Solution : Using Gaussian elimination with the augmented matrix.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \\ 10 & 11 & 12 & 0 \end{array}\right) \xrightarrow[-10E1+E4]{\substack{-4E1+E2 \\ -7E1+E3}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & 0 \\ 0 & -9 & -18 & 0 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{3}E2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -6 & -12 & 0 \\ 0 & -9 & -18 & 0 \end{array}\right) \xrightarrow[9E2+E2]{6E2+E3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Therefore

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ x_2 + 2x_3 &= 0 \end{aligned}$$

and setting x_3 = k gives $x_2 = -2k$

$x_1 = -2x_2 - 3x_3 = 4k - 3k = k$

So we have

$$x_1 = k$$
$$x_2 = -2k$$
$$x_3 = k$$

38. Show that the equations

x + 2y – z = 3,
3x – y + 2z = 1,
2x – 2y + 3z = 2,
x – y + z = 1 are consistent and solve them.

Solution : The given system of equations is equivalent to the single matrix equaion

$$AX = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ -1 \end{bmatrix} = B.$$

The augmented matrix [A B] = $\begin{bmatrix} 1 & 2 & -1 & : & 3 \\ 3 & -1 & 2 & : & 1 \\ 2 & -2 & 3 & : & 2 \\ 1 & -1 & 1 & : & -1 \end{bmatrix}$.

Performing $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - R_1$, we get

$$[A \quad B] \sim \begin{bmatrix} 1 & 2 & -1: & 3 \\ 0 & -7 & 5: & -8 \\ 0 & -6 & 5: & -4 \\ 0 & -3 & 2: & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1: & 3 \\ 0 & -1 & 0: & -4 \\ 0 & -6 & 5: & -4 \\ 0 & -3 & 2: & -4 \end{bmatrix} \text{ by } R_2 \to R_2 - R_3$$

$$\sim \begin{bmatrix} 1 & 2 & -1: & 3 \\ 0 & -1 & 0: & -4 \\ 0 & 0 & 5: & 20 \\ 0 & 0 & 2: & 8 \end{bmatrix} \text{ by } R_3 \to R_3 - 6R_2,\ R_4 \to R_4 - 3R_2,$$

$$\sim \begin{bmatrix} 1 & 2 & -1: & 3 \\ 0 & -1 & 0: & -4 \\ 0 & 0 & 1: & 4 \\ 0 & 0 & 1: & 4 \end{bmatrix}, \text{ by } R_3 \to \frac{1}{5} R_3,\ R_4 \to \frac{1}{2} R_4$$

$$\sim \begin{bmatrix} 1 & 2 & -1: & 3 \\ 0 & -1 & 0: & -4 \\ 0 & 0 & 1: & 4 \\ 0 & 0 & 0: & 0 \end{bmatrix}, \text{ by } R_4 \to R_4 - R_3.$$

Thus the matrix [A B] has been reduced to Echelon form. We have rank [A B] = the number of non-zero rows in this Echelon form = 3. Also

$$A \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We have rank A = 3. Since rank [A B] = rank A, therefore the given equations are consistent,. Since rank A = 3 = the number of unknowns, therefore the given equations have unique solution. The given equations are equivalent to the equations

x + 2y – z = 3, –y = –4, z = 4.

These give z = 4, y = 4, x = –1.